

Topology and Uniqueness of Higher Dimensional Black Holes

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We review recent results concerning general properties of higher dimensional black holes. The topics selected with particular focus are those concerning topology, symmetry, and uniqueness properties of asymptotically flat vacuum black holes in higher dimensional general relativity.

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§1. Introduction

The black hole uniqueness theorem in 4-dimensions is a triumph of classical general relativity, implying that a tremendous number of black holes existing in our

observable universe can be described accurately by the Kerr metric, which possesses only two parameters. In the course of complete proof of the uniqueness theorem, there have appeared a number of remarkable results, each of which itself reveals physically an important property of black holes, such as those concerning topology and symmetry.¹⁾ These results also give us deep insights into thermodynamic aspects of black holes.²⁾

It is of great interest to consider generalizations of a number of theorems established for 4-dimensional black holes to higher dimensional case. One might expect that such a generalization could straightforwardly be done by merely replacing “4” with a general number “ D .” However, as is by now well-known, the discovery of the black ring solution in 5-dimensions³⁾ (as well as a large variety of exact solutions discussed in Chapter 1 and 4) has drastically changed our view of the issue, highlighting that the uniqueness theorem no longer holds as it stands in higher dimensions.

In this chapter we shall review general properties of higher dimensional black holes, attempting to clarify which properties of 4-dimensional black holes can be straightforwardly generalized to higher dimensions and which properties hold only in 4-dimensions. There have already been a number of theorems for higher dimensional black holes. This review is, however, not intended to cover the whole relevant subjects or supply a complete list of existing literature. Rather, we will focus on some specific topics relevant to topology and uniqueness/non-uniqueness feature of asymptotically flat vacuum black holes and describe some key ideas and methods for obtaining the results in higher dimensions.

In the next section we shall first recapitulate how to describe global structure of black hole spacetimes in general dimensions. Then we discuss topological aspects of apparent and the event horizon of dynamical black holes. In section 3, we consider stationary black holes in higher dimensional general relativity. After briefly summarizing a few critical steps in the proof of black hole uniqueness theorems in 4-dimensions, we shall discuss how and to what extent each of the steps can be generalized to higher dimensional setting. In particular, we review recent results concerning possible restrictions on the horizon topology and also an enhancement of Killing symmetries, called the rigidity theorem, in higher dimensions. In section 4, we review uniqueness theorems for static black holes in higher dimensions. Uniqueness theorems for asymptotically flat, stationary rotating, vacuum black holes are discussed in detail in section 5. Section 6 is devoted to summary.

Notations and Conventions

In this chapter, we mainly use the abstract index notation for tensor fields on a spacetime, where each slot for tangent or cotangent field is denoted by a lower-case latin index: a, b, c, \dots . In section 4, a tensor field is written in terms of its components, where upper-case latin indices M, N, \dots run from 0 to $D - 1$, and lower-case latin indices i, j, \dots run from 1 to $D - 1$. In section 5, we mainly treat the 5-dimensional spacetimes. There, a tensor field is mainly written in terms of its components, where lower-case latin indices i, j, \dots run from 1 to 2 and upper-case latin indices I, J, \dots run from 3 to 5.

We use the natural unit where the speed of light c and the reduced Planck constant \hbar are set to unity.

§2. General properties of higher dimensional black holes

We begin with noting that the uniqueness feature of a stationary black hole is related to its thermodynamic aspects in the sense that a thermodynamically equilibrium system can completely be characterized by a small number of state parameters.²⁾ The idea of black hole thermodynamics has originated from Bekenstein’s interpretation⁴⁾ of Hawking’s area theorem⁵⁾ as the 2nd law of thermodynamics, as well as from the black hole mechanics due to Bardeen, Carter and Hawking.⁶⁾ While the latter concerns a stationary equilibrium configuration, the former involves a dynamical process concerning the total area of all black holes in the universe. Therefore, before going into discussion of stationary black holes, in this section we shall discuss general circumstances that can include some dynamical processes such as a formation and evolution of black holes, to which the area theorem becomes relevant.

2.1. Global structure and area theorem

First of all, in order to define an isolated black hole in general context, one needs to introduce a suitable notion of “infinity” and associated asymptotic structure. In 4-dimensions this is usually, and elegantly, done in the conformal framework, in which an unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$, conformally isometric to our physical spacetime (M, g_{ab}) in $M \cap \tilde{M}$, plays a role. A desired notion of infinity and asymptotic flatness can be defined by specifying the behavior of the conformal metric \tilde{g}_{ab} near a conformal null boundary $\mathcal{I} = \partial\tilde{M}$. If one further imposes an additional condition that every maximally extended null geodesic in M has past and future endpoints on null boundary \mathcal{I} in \tilde{M} , then \mathcal{I} is divided into disjoint sets of the future and past null infinity, \mathcal{I}^+ and \mathcal{I}^- . Such a spacetime is called *asymptotically simple*. A spacetime (M, g_{ab}) is said to be *weakly asymptotically simple at null infinity* if $(\tilde{M}, \tilde{g}_{ab})$ has a neighborhood of \mathcal{I} which is isometric to a neighborhood of \mathcal{I} for some asymptotically simple spacetime. The notion of *strong asymptotic predictability* is then defined such that the closure of $M \cap J^-(\mathcal{I}^+)$ is contained in a globally hyperbolic open subset of \tilde{M} . A black hole region \mathcal{B} is defined as the complement of $J^-(\mathcal{I}^+)$ and the future event horizon H as the boundary of \mathcal{B} in M . As such, H is a null hypersurface ruled by null geodesics. Since \mathcal{B} is a future set, every null geodesic generator of H is future inextendible, but in general admits a past end point. With the set of these definitions, general properties of 4-dimensional black holes are studied by using the global method, which consists of a number of general results concerning causal structure, behavior of causal geodesic congruence, etc, as in Refs. 7), 8).

A key equation for the global method is the Raychaudhuri equation for causal geodesics, which together with certain energy condition, governs the occurrence of (a pair of) conjugate points. The structure of the Raychaudhuri equation is unchanged in higher dimensions as far as (higher dimensional version of) general relativity is

considered; it takes the form for, e.g., a surface orthogonal null geodesic congruence in D -dimensions,

$$\frac{d}{d\lambda}\theta = -\frac{1}{D-2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} - R_{ab}k^ak^b, \quad (2.1)$$

with $k^a = (d/d\lambda)^a$, θ , $\hat{\sigma}_{ab}$ being the tangent of a null geodesic with affine parameter λ of the congruence, its expansion and the shear. Therefore, once well-defined notions of asymptotic flatness at null infinity and strong asymptotic predictability are formulated in higher dimensions, one can apply general results in 4-dimensions—more specifically, Propositions and Theorems in Section 12.2 of Wald⁸⁾—to higher dimensions. Note that the predictability is needed, in particular to show the area theorem without demanding that null geodesic generators of the event horizon be complete. Note also that in order for a weakly asymptotically simple spacetime in D -dimensions to be consistent with the asymptotic simplicity under the additional condition on maximally extended null geodesics mentioned above, each component of \mathcal{I} has to be topologically $\mathbf{R} \times S^{D-2}$. This, combined with the topological censorship,⁹⁾ ensures that the domain of outer communication is simply connected.

For $D = \text{even}$ -spacetime dimensions, there exists a stable notion of conformal null infinity and weak asymptotic simplicity [see Ref. 10) for definition], and therefore there is no obstruction to apply general results [i.e., those in Section 12.2 of Ref. 8)] to higher-even-dimensional spacetimes.

However, when spacetime dimension is odd, one needs to be more careful; The conformal method for defining null infinity would not in general work since the unphysical metric fails to be smooth at conformal null infinity \mathcal{I} when radiation is present around \mathcal{I} .¹¹⁾ This is essentially due to the fact that the leading fall-off behavior of gravitational radiation near null infinity is in proportion to a half-integer power of the conformal factor, Ω , when D is odd. Therefore, for the case of odd spacetime dimensions, we need to formulate a sensible definition of null infinity that can be used to define asymptotic flatness and some equivalent notion of the strong asymptotic predictability [see Ref. 12) for such an attempt to define asymptotic flatness in 5-dimensions without using the conformal method]*).

In the following when we discuss a black hole in a dynamical, non-stationary spacetime, we simply assume that a sensible definition of infinity and asymptotic flatness, equivalent to \mathcal{I} and weak asymptotic simplicity above, are formulated, and we just use the same symbol \mathcal{I} to denote thus defined infinity, even if the spacetime dimension is odd and gravitational radiation is present near infinity. This should be kept in mind when, e.g., a topology changing process of cross-sections of the event horizon is considered, since such a phenomenon can only occur in non-stationary, dynamical spacetime where gravitational radiation is likely to generate.

With the above caveat concerning definitions of null infinity and the predictabil-

*) One may also want to consider non-asymptotically flat spacetimes, such as asymptotically Kaluza-Klein spacetimes. For that case, we would not be able to use the standard conformal approach to defining null infinity since the compactified dimensions shrink to a single point by the conformal transformation. We again need to formulate a suitable definition of infinity, presumably by dealing with the physical spacetime metric and its asymptotic expansion. Note however the stationary case discussed below.

ity, the standard proof of the theorem [Prop. 9.2.8 in Ref. 7), Prop. 12.2.2 in Ref. 8)] that under the null convergence condition, an apparent horizon (see below) is contained in the black hole region is straightforwardly generalized to arbitrary dimensions. In particular, the standard proof of the Hawking’s black hole area theorem⁷⁾ [Theorem 12.2.6 in Ref. 8)] is generalized to arbitrary higher dimensions.

2.2. Apparent horizon

For some purposes, instead of dealing with the event horizon, one is more interested in an *apparent horizon* which, in a sense, defines a black hole region in a local manner and plays a role in particular in numerical studies [See Chapter 9]. This notion can be straightforwardly generalized to higher dimensional case as discussed below.

Let us consider a partial Cauchy surface, S , in D -dimensional spacetime M , which is assumed to be an $(D - 1)$ -dimensional connected hypersurface smoothly embedded in M . A closed connected $(D - 2)$ -surface, T , smoothly embedded in S is called a *trapped surface* if the expansion of the congruence of outgoing light rays orthogonal to T is non-positive at each point of T . This definition of the trapped surface is well-defined only when the notion of “outgoing” has a definite meaning. The asymptotic flatness and the orientability of S and T do not give the unique definition of ‘out direction’ on T . In many cases, it is assumed that S is also simply connected. In this case, the out direction of T can be defined in terms of the \mathbf{Z}_2 -intersection numbers of curves from T to the spatial infinity. However, since the simple connectedness of S might be too restrictive, it is worth giving another example of a condition that determines the out direction of T without an ambiguity. Possible such conditions are that S is orientable and that T separates S into two disconnected parts. In other words, it is required that $S \setminus T = S_{\text{in}} \cup S_{\text{out}}$ and $S_{\text{in}} \cap S_{\text{out}} = \emptyset$ hold, where S_{out} is determined by the property that it contains a neighborhood of the spatial infinity. This clearly ensures that T is orientable and defines the out direction of T in an obvious way. This condition also applies even when S is not simply connected.

Under the above conditions, the inner region S_{in} with respect to a trapped surface T is called the *inside region* $S_{\text{in}}(T)$ of T . The inside region of T will be a closed subregion of S , whose topological boundary in S consists of T . Then, the *trapped region* of S is defined to be the topological sum of $S_{\text{in}}(T)$ over all possible T . The trapped region might not be a closed region or a smooth region of S . For simplicity, we however only consider the case where the trapped region is a smooth closed region of S . Then, the apparent horizon on S is defined to be the topological boundary of the trapped region. It turns out that the apparent horizon on S is a *marginally outer trapped surface*, or in other words, that the expansion of the congruence of the outgoing light rays orthogonal to the apparent horizon is zero everywhere on it.⁷⁾

Now let us consider topological aspects of apparent horizons. Hawking¹³⁾ has shown under the dominant energy condition that the apparent horizon in 4-dimensional spacetime must be diffeomorphic to a 2-sphere or possibly to a 2-torus. Hawking’s proof of the horizon topology theorem takes two steps: First (i) it is derived under

the dominant energy condition that

$$\oint_{\Sigma} \mathcal{R} dS \geq \oint_{\Sigma} 8\pi G T_{ab} l^a (l^b + n^b),$$

where Σ denotes the apparent horizon, \mathcal{R} is the scalar curvature with respect to the Riemannian metric induced on Σ , T_{ab} is the stress-energy tensor, l^a and n^a respectively are the outgoing and the incoming future pointing null vector field orthogonal to Σ . The dominant energy condition requires that $T_{ab} l^a (l^b + n^b) \geq 0$ holds everywhere. Therefore it follows

$$\oint_{\Sigma} \mathcal{R} \geq 0. \quad (2.2)$$

Then (ii) it is appealed to the Gauss-Bonnet theorem, which says that the left hand side of the above inequality is $2\pi\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic number of Σ . It immediately follows that Σ must be topologically 2-sphere or 2-torus. The case where the equality in (2.2) holds, if exists, seems to describe an unstable configuration. For example, it fails if we add arbitrary small positive cosmological constant. Therefore, the toric apparent horizon seems implausible, though we need further technical assumptions to exclude it. [See Ref. 14), for the issue.] In any case, for simplicity, we just assume that the strict inequality of (2.2) holds in the following argument.

Although the inequality (2.2) in Step (i) holds true also in general spacetime dimensions $D \geq 4$ without modifications other than that Σ is $(D-2)$ -dimensional as shown in Ref. 15), the Gauss-Bonnet theorem in Step (ii) can apply only in 4-dimensions. Noting this fact, Galloway and Schoen^{16),14)} have shown, as a natural generalization of Hawking's topology result mentioned above to higher dimensions, that an apparent horizon must admit a Riemannian metric of positive scalar curvature. This in general restricts possible topological or differentiable structure of the apparent horizons. Now the problem has reduced to a purely geometric argument, and we can apply the mathematical results in the standard differential geometry. For example, in 5-dimensions, Σ must be either 3-sphere with possibly identifications, or $S^2 \times S^1$, or a finite connected sum of them. Of course, an apparent horizon in 5-dimensional spacetime must be homeomorphic to one of them.

Thus, we have many possibilities for the topology of the apparent horizon. Since it is known that a spatial section of the event horizon in a stationary spacetime coincides with the apparent horizon, this result indicates that there might be a rich variety for the final equilibrium configuration of black holes in higher dimensions. In fact, such an example has been explicitly constructed by Emparan and Reall.³⁾ They have found a black hole solution whose black hole horizon is diffeomorphic with $S^2 \times S^1$ in 5-dimensional Ricci flat spacetimes.

It is interesting to note that by applying a similar argument used in the derivation of (2.2) to asymptotically (locally) anti-de Sitter black holes, one can obtain a topology dependent lower bound for the area, hence entropy, of the black hole^{17),18)} [see also Refs. 19), 20), 21)].

2.3. Topological structure of dynamical black hole horizons

Next, let us briefly see the dynamical aspects of black hole horizons. We focus on the time evolution of the topological structure of event horizons. The future event horizon H is generated by null geodesics without a future end point. The set consisting of all the points where two or more null geodesic generators of H intersects is called the *crease set* F of H . Each null geodesic generator of H has a past end point on the closure \overline{F} of F , or else it does not have a past end point. We here consider black holes which is created within finite past, or in other words, we only consider the case where any geodesic generator has a past end point on \overline{F} in the following. The event horizon H may consists of several connected components, which might occur, for example, when the final state is represented by one of the Majumdar-Papapetrou solutions to the Einstein-Maxwell equations describing equilibrium states of several extremely charged black holes. However, it is enough to consider the case where H consists of one connected component, as we assume here, for generalization to the case with many black holes is straightforward. From the definition of F , it immediately follows that the crease set F is an acausal set in H ,²²⁾ that is any two points in F are causally separated.

More stringently, the following theorem holds in D -dimensional spacetimes: if the partial Cauchy surface is \mathbf{R}^{D-1} and the spatial section of H is homeomorphic with the $(D-2)$ -sphere at sufficiently late times, the crease set F is a contractible space. In particular, it immediately follows that F is arcwise connected. This gives a simple picture for the time evolution of event horizons. In fact, it is easy to show that there is a time slicing of M in which a black hole is created at a point, and the spatial section of H is homeomorphic with $(D-2)$ -sphere in subsequent times.

Another interesting viewpoint of Siino²²⁾ is that the number of black holes is a gauge dependent notion. The event horizon is clearly a gauge invariant object, defined only by the terminology in the causal structure of spacetimes. However, whenever we count the number of black holes, we first prepare appropriate time slicing of M to do it, which is clearly a gauge dependent procedure. For example, let us consider the head on collision of two spherical black holes into one. This situation will realize in an axisymmetric configuration, where the axis of symmetry is the straight line along which two black holes take trajectories. Then, we might expect that the spatial section of H before the collision consists of two connected components. In fact, it is not always the case. In this situation, the crease set F will be a topological line segment in the world surface of the symmetric axis. Since F is an acausal set, it is just a spacelike line segment. Now, we can take a partial Cauchy surface S , such that entire F is included in S . Then, if we consider a time slicing of M including S as a time slice, we will not see disconnected black holes at any time. In other words, this spacetime can be interpreted as just a creation of a black hole. Furthermore, since F is a spacelike line segment as noted above, we can also choose a partial Cauchy surface S' , where S' intersects F transversally at many points. Then, we will conclude that there are more than two black holes on S' .

A better theorem has been obtained by Ida.²³⁾ The event horizon H is an $(D-1)$ -dimensional topological submanifold of M , which follows from the fact that

it is the boundary of a causal past $J^-(\mathcal{I}^+)$ in M .⁷⁾ Note that it may not be smoothly embedded in M , but it has a wedge-like structure at \overline{F} in general. The following theorem gives a relationship between the topological structure of F and that of the topological manifold H : The crease set F in H is a deformation retract of H . Then, it immediately follows that the homotopy type of F and H coincides. In particular, the crease set must be arcwise connected, which has already been noted by Siino.²²⁾ However, this theorem also applies to homotopy types of higher order. For example, F must be simply connected if H is.

It also holds²³⁾ that the final topology of the spatial section of the event horizon is diffeomorphic with the ϵ neighborhood of F in H , where ϵ neighborhood is defined by introducing a Riemannian metric on H determined in terms of the differentiable structure on M (of course, it is not the induced metric). In this sense, the topology of the black hole is determined by the crease set F .

Next, let us discuss several forbidden processes in topology change of black holes. A well known is the no bifurcation theorem of black hole due to Hawking,⁷⁾ which says that a black hole does not evolve into several black holes by bifurcating. More precisely, it states that $J^+(\mathcal{B}_0) \cap S_1$ is arcwise connected, where \mathcal{B}_0 denotes a connected black hole region on a partial Cauchy surface S_0 [so $\mathcal{B}_0 \subseteq \mathcal{B} \cap S_0$], and S_1 is a partial Cauchy surface at later time, that is entirely lying within $I^+(S_0)$. This gives one of the most basic forbidden processes for black holes. Since it seems that there are more possibilities for black hole topology in higher dimensional spacetimes, one might expect that there might be other forbidden processes than this. This problem is considered in Ref. 24) in the language of the Morse theory. First, note that the topology of a black hole does not change at sufficiently late times, where time slices do not intersect the crease set F , since the null geodesic generators of the event horizon naturally defines the diffeomorphism between the black hole horizons on different time slices. In other words, the black hole horizon can change its topology only at the instant when the time slice intersects the crease set F . From this viewpoint, the topology change of black holes can be regarded as a local process occurring in a small neighborhood of F . In order to apply the Morse theory to the event horizon, we first introduce a smoothed event horizon \tilde{H} , which is a differentiable manifold smoothly embedded in M , such that this smoothing procedure preserves the topological structure of black hole horizon on each time slicing. Furthermore, we consider a situation where the time function t induced on the smoothed event horizon \tilde{H} gives a Morse function on \tilde{H} , that each critical point of t , defined as such point where $\partial_i t = 0$ holds, where the partial derivatives are those on \tilde{H} , is a regular critical point, where the $\det(\partial_i \partial_j t) \neq 0$ holds. This technical assumption is not always justified, although it is always possible for any black hole spacetime to find such time slicings that the above operations are justified. Once the time function t on \tilde{H} is recognized as a Morse function on \tilde{H} , the topology change of the black hole horizon can be seen as a local process occurring at a critical point of t on \tilde{H} . This allows us only to consider the evolution of a local black hole region around a critical point, which can in fact be regarded as a process in the Minkowski spacetime. In this way, it turns out that there are many forbidden processes for the topology change of black hole horizons.

Here, we provide an example among these. Let us consider the process where an initial black hole horizon homeomorphic with S^{D-1} change its topology into $S^{D-2} \times S^1$. There are topologically distinct two possible processes for this to occur at a specific instant of time. One possibility is such that a pair of horns grows, one from the North pole, and the other from the South pole of the sphere, and then merges each other at a point. The other possibility is that the sphere punctures in such a way that the North pole and the South pole meet each other from the inside of the sphere. The former process can actually occur, but the later turns out to be a forbidden process.

Finally, we consider the topology of crease set of the event horizon in 5-dimensional spacetimes. In this case, the topology of the black hole at late times is explicitly known. Note that if the black hole horizon in final equilibrium state is diffeomorphic with a 3-manifold N , the event horizon H is diffeomorphic with the interior of a 4-manifold bounded by N . Furthermore, H must be embedded in the partial Cauchy surface. They can be seen by noting that H is an acausal set and that a time-like vector field without zero, which always exists in any time orientable Lorentzian manifold, naturally defines a diffeomorphism of H into a portion of a partial Cauchy surface.

First, let the partial Cauchy surface S be \mathbf{R}^4 . Then, N must be S^3 , $S^2 \times S^1$, or a connected sum $\#_m(S^2 \times S^1)$. The lens space black hole is not realized in this case, for it cannot be embedded in \mathbf{R}^4 . If the crease set F is a point, then H is an open 4-disk and N is a 3-sphere. If F is a circle, then H is an open solid torus and N is $S^2 \times S^1$. The crease set for $N \simeq \#_m(S^2 \times S^1)$ is given by a chain of m circles. The chain of circles is here defined as follows. Let $\{C_i\}$ ($i = 1, 2, \dots, m$) be a sequence of circles. Then the chain of m circles is the space $\bigcup_i C_i$ with identification of a point in C_i and a point in C_{i+1} for each $i = 1, \dots, m-1$.

To realize the lens space black holes, we must consider the partial Cauchy surface that is not \mathbf{R}^4 . A simplest example is obtained when H is a 2-dimensional real vector bundle over the 2-sphere. The topology of such spaces is classified by the Euler number $p \in \mathbf{Z}$, which corresponds to the Dirac monopole number of the associated principal $U(1)$ bundle. Hence we denote it by E_p . Such an event horizon $H \simeq E_p$ emerges when the crease set is the zero section of E_p , that is the 2-sphere. The vector bundle E_p is clearly bounded by S^1 bundle over the 2-sphere, that gives the final topology of the black hole horizon. The vector bundle E_0 is just a product space $S^2 \times \mathbf{R}^2$, which is bounded by $S^2 \times S^1$. Thus the black ring can also emerge from $F \simeq S^2$. The vector bundle E_1 is bounded by the 3-sphere. This is known as the Hopf fibration of S^3 . Hence, the spherical black hole can also emerge from $F \simeq S^2$. The Kaluza-Klein black hole discussed in Ref. 25) belongs exactly to this type. For general p , ($|p| \geq 2$), the boundary of E_p becomes the lens space $L(p, 1)$. Thus, the lens space black hole of this special type can emerge from $F \simeq S^2$. In fact, the black lens spacetime in Ref. 23) corresponds to $H \simeq E_2$. General black lens horizon $\simeq L(p, q)$ is modelled by H , which is a plumbing of a sequence $\{E_{p_i}\}$, namely the chain of vector bundles in an appropriate sense. This is discussed by Ida,²³⁾ in which it is shown that the 5-dimensional Kastor-Traschen spacetime²⁶⁾ includes black lens spacetime with the event horizon consisting of plumbing of m vector bundles E_2 , to

result in $L(m+1, 1)$ black lens horizon.

§3. Stationary black holes in higher dimensions

Now we turn to stationary black holes in a smooth, strongly causal spacetime (M, g_{ab}) . By *stationary* we mean that there exists a Killing vector field t^a whose orbits are complete everywhere in the spacetime and are timelike at least at large distances. The completeness is important in the following arguments, as otherwise t^a would fail to generate an isometry group, ϕ_t . In this case, there is no problem to define the notion of asymptotic flatness at null infinity of the standard topology $\mathcal{I} \approx \mathbf{R} \times S^{D-2}$ in both even and odd dimensions within the standard conformal framework. For more general cases including asymptotically Kaluza-Klein spacetimes, one may apply the method using an *asymptotically flat end*, S_{ext} , of a partial Cauchy surface S , on which the induced metric and extrinsic curvature satisfy some appropriate fall-off conditions. [See Refs. 27), 28), 29) for details.] The one-parameter group of diffeomorphism of t^a defines an asymptotically flat exterior region by $M_{\text{ext}} = \cup_{t \in \mathbf{R}} \phi_t(S_{\text{ext}})$. The black hole region is then $\mathcal{B} = M \setminus J^-(M_{\text{ext}})$. One can choose a cross section Σ of $H = \partial\mathcal{B}$ so that t^a is everywhere transverse to Σ on $H \cap I^+(M_{\text{ext}})$. By Lie-dragging Σ over H along the orbits of t^a , one can define a foliation, $\phi_t(\Sigma)$. Then, using the Raychaudhuri equation, (2.1), the vacuum Einstein equations, and the same arguments as used to prove the area theorem, one can show that the expansion and shear of the null geodesic generators of the event horizon vanish as asserted in Prop. 9.3.1 of Ref. 7). It then follows that all cross-sections of the event horizon of a stationary black hole are isometric, as far as they are chosen in the manner just mentioned above. [Compare with the non-stationary case discussed in the previous section.]

It may be instructive to start with briefly recapitulating basic steps of the uniqueness proof for asymptotically flat, stationary vacuum black holes in 4-dimensions and then to describe attempts to generalize those steps to higher dimensions. The proof in 4-dimensions goes roughly as follows:

(i) *Topology theorem:* The first step in the proof is to show that each connected component of a cross section of the event horizon of an asymptotically flat, stationary black hole must have spherical topology. This was first shown by Hawking¹⁾ [see also Prop. 9.3.2 of Ref. 7), and the previous argument in subsection 2.2], in which a variational argument, the dominant energy condition, and the Gauss-Bonnet theorem are used. A stronger proof was later given by Chruściel and Wald³⁰⁾ by using the topological censorship,⁹⁾ the null energy condition, and cobordism theory. The topological censorship, which requires the weak energy (or null convergent) condition, is used to show the simple connectedness of the domain of outer communications. Then basic result of cobordism theory yields that a simply connected Riemannian 3-manifold with a boundary S^2 at one end (at infinity) must have S^2 at another end (at horizon).

(ii) *Rigidity theorem:* The next step is to show that a stationary black hole is either static or axisymmetric. One can show, by using the result of topology theorem (i)

that (1) there exists a Killing field K^a which is normal to H , hence $K^a K_a = 0$ on H , and furthermore that (2) if the black hole is rotating, there must exist another independent Killing vector field ϕ^a , which commute with K^a and generates an isometry group $U(1)$. Thus, the spacetime isometry group is at least $\mathbf{R} \times U(1)$. [See Prop. 9.3.6 of Ref. 7). See also Refs. 31), 32).] (1) establishes that the event horizon is a *Killing horizon*. An immediate consequence is the constancy over H of the surface gravity, κ , defined by $K^b \nabla_b K^a = \kappa K^a$ on H . Since κ defines the Hawking temperature, this establishes the basis of the 0th-law of black hole thermodynamics. (2) establishes that the spacetime, when rotating, is *axisymmetric*. Since it implies that the generators of the event horizon are *rigidly* rotating with respect to infinity, it is called the rigidity theorem.

When t^a is null on H , i.e., non-rotating, we appeal to the staticity theorem of Ref. 33) to show that the spacetime is static. Then, we apply the theorem of Israel³⁴⁾ to show the uniqueness of such a solution. We will discuss this case and its higher dimensional generalizations in detail in Section 4.

(iii) *Non-linear sigma model and global divergence identity*: The next step is, by using the two isometries obtained in Step (ii), we reduce the Einstein equations to a certain type of non-linear sigma model on a 2-dimensional base space. We see that the target space of the sigma-model is homogeneous and non-compact, thus having a non-positive sectional curvature. In fact it is $SL(2, \mathbf{R})/SO(2) \cong \mathbf{H}^2$ for the vacuum case. The reduction is achieved with the help of the Weyl-Papapetrou coordinates. We also construct a (Mazur’s) divergence identity [see also Bunting’s in section 5 for non-homogeneous case], expressed in terms of a coset matrix. Note that this is a global identity as the target space is homogeneous.

(iv) *Boundary value analysis*: We show by using the global divergence identity that given boundary conditions that correspond to specifying a set of asymptotic conserved charges, the solutions are uniquely determined.

Technical details of the uniqueness proof are found in literature 28), 35). For the rest of this section we shall review how the first two steps, (i) and (ii), are generalized to higher dimensions. Higher dimensional generalizations of the last two steps, (iii) and (iv), will be discussed in more detail in Section 5.

3.1. Topology theorems in higher dimensions

3.1.1. Topology of cross sections of the event horizon

As mentioned above, all cross-sections Σ of the event horizon H of a stationary black hole are isometric—thus slice independent—in $I^+(M_{\text{ext}})$, and this property holds true in higher dimensional case as well. Furthermore an apparent horizon of a stationary black hole coincides with the event horizon. Therefore the topology theorem for an apparent horizon discussed in the previous section^{13), 16), 14)} applies to the event horizon:

Theorem 3.1: [Reference 16), 14)] *Cross sections of the event horizon in stationary black hole spacetimes obeying the dominant energy condition are of positive Yamabe type.*

This implies that cross sections Σ must be able to carry a metric of positive scalar curvature. In particular, the case of toroidal cross sections has been excluded.¹⁴⁾ Accordingly, in 5-dimensions, Σ must be a connected sum of S^3 , $S^2 \times S^1$, and some quotient space of S^3 , including a Lens space.

A higher dimensional generalization of the proof of Chruściel and Wald,³⁰⁾ in which the topological censorship and cobordism is used, has also been discussed by Helfgott, Oz and Yanay.¹⁵⁾ However, the topological constraint thus obtained is less restrictive than that of Refs. 16), 14) above.

Further topological restrictions can be obtained when a stationary black hole admits extra symmetries. Suppose a D -dimensional stationary black hole spacetime admits $(D - 3)$ -rotational isometries. (Note that for $D \geq 6$, this assumption is no longer compatible with the standard notion of asymptotic flatness with S^{D-2} spatial infinity. It is, instead, relevant to asymptotically Kaluza-Klein spacetimes.) It is known that such a stationary, multi-axisymmetric spacetime possesses a set of invariants—called the *rod-structure* of Harmark^{36), 37)}—which consists of a set of positive numbers and vectors living in a boundary segment of the 2-dimensional factor space of M by the isometry and defines the structure of the ‘zero sets’ of the Killing vector fields. The rod-structure encodes, especially, information about the horizon topology. Given a rod on the boundary segment, the associated positive number corresponds to the invariant length of the rod, and the vector associated with the rod (apart from that corresponding to the horizon) specifies which linear combination of the (rotational) Killing vector fields vanishes in the rod. [See Chapter 4 for details of the rod-structure. See also Ref. 38) for earlier work on the rod-structure considered in static spacetimes]. A refined version of the rod-structure—called the *interval structure*—has been proposed in Refs. 39), 40). In particular, it has been made clear that the determinant of the matrix made of two integer valued vectors associated with adjacent rods characterizes the horizon topology. For example, it has been shown that for a 5-dimensional asymptotically flat, stationary vacuum black hole with two axial symmetries, each connected component of the horizon cross section must be topologically either S^3 , $S^2 \times S^1$, or a Lens-space. The concrete relation between the possible horizon topology and the interval structure has been given [see Prop. 2 of Ref. 39)]. This result has been generalized to a more general case of stationary asymptotically Kaluza-Klein black holes with $(D - 3)$ rotational isometries.⁴⁰⁾

The rigidity theorem states that any stationary rotating black hole spacetime must be axisymmetric. This holds true in higher dimensions as well [see next subsection] but it guarantees the existence of only a single $U(1)$ isometry, irrespective of the number of spacetime dimensions. Therefore there could exist a stationary black hole solution in higher dimensions that has precisely a single $U(1)$ isometry, as conjectured by Reall.⁴¹⁾ Having only a single $U(1)$ isometry, we would have less restrictive constraint on the horizon topology than the case of having more than one axial symmetries, say $U(1)^{D-3}$ ($D \geq 5$) discussed above, but more restrictive constraint than what is implied by merely knowing that Σ is of positive Yamabe type.^{16), 14)} It has been shown in Ref. 42) that the horizon cross section of 5-dimensional asymptotically flat, stationary vacuum black holes can be either connected sum of Lens spaces and

handles ($S^1 \times S^2$), or the quotient of S^3 by certain finite subgroups of isometries with no handles. The latter horizon manifold includes Prism manifolds, quotients of the Poincare homology sphere, and a various Seifert fibred spaces over S^2 [see Table 1 of Ref. 42) for the list of possible horizon topologies].

The notion of the rod-structure—which requires $(D - 2)$ mutually commuting isometries—has been generalized to the *domain structure*,⁴³⁾ which applies to the case of any spacetime dimensions and any numbers of Killing vector fields. Thus, in particular, the domain structure can be used to characterize a stationary black hole geometry with fewer rotational symmetries than $(D - 3)$ rotational symmetries.

3.1.2. Topology of the domain of outer communications

Compared with the horizon topology, perhaps less appreciated is the topology of the domain of outer communications. The topological censorship^{9), 44), 45)} states that any curve in the domain of outer communication with endpoints in the asymptotic region M_{ext} can be deformed to a curve entirely within M_{ext} . Therefore, for a black hole in an asymptotically flat spacetime in the standard sense, the domain of outer communication must be simply connected. However, in higher dimensions, the simple connectedness itself does not completely determine the topology of a partial Cauchy surface S in the domain of outer communication, though it is likely to be \mathbf{R}^{D-1} minus a compact manifold \mathcal{B} that describes the black hole region on S .

In 5-dimensions, it has been shown in Ref. 40) by using the results of Ref. 46) that when a stationary, rotating black hole spacetime admits two axial isometries $U(1) \times U(1)$, the domain of outer communication has topology $\mathbf{R} \times S$, where for some $n, n' \in \mathbf{N}$,

$$S \cong \left(\mathbf{R}^4 \# n \cdot (S^2 \times S^2) \# n' \cdot (\pm CP^2) \right) \setminus \mathcal{B}, \quad (3.1)$$

where \mathcal{B} denotes a compact manifold with boundary $\partial\mathcal{B} = \Sigma$, an intersection of a partial Cauchy surface S and the black hole region \mathcal{B} . It has also been shown in Ref. 40) that as in the case of the horizon topology, the topology of domain of outer communication can be completely specified in terms of the interval structure. Note also that if the black hole is non-rotating, then it is static due to the result of Ref. 33). Then, the solution is shown to be isometric to the 5-dimensional Schwarzschild spacetime.^{47), 48), 49)} [See Section 4.] Therefore $S \cong \mathbf{R}^4 \setminus \mathcal{B}$.

For 5-dimensional stationary black holes with only one $U(1)$ isometry, in accord with the rigidity theorem, one might expect to have less restrictive constraints on the possible topology, as in the even horizon case. However, it is shown by Ref. 42) that the topological constraint on domain of outer communication of this less symmetric case is essentially the same as the case with two rotational isometries $U(1) \times U(1)$.

3.2. Rigidity theorem in higher dimensions

A higher dimensional generalization of the rigidity theorem has been made by Hollands, Wald and one of the present authors,⁵⁰⁾ as well as by Moncrief and Isenberg.⁵¹⁾ In this subsection we shall review the results obtained in Ref. 50).

Hawking’s proof of the rigidity theorem relies in an essential way upon the fact

that the spacetime dimensions is 4, and for this reason, it is a priori not at all obvious whether the rigidity theorem can be generalized to higher dimensional case. Let us start with a brief sketch of the rigidity proof and see how the spacetime dimensionality enters the proof. Since the stationary Killing vector field t^a generates a one-parameter group of isometries, it must be tangent to the event horizon, H . When t^a is not normal to H (i.e., not tangent to the null geodesic generators of H), the black hole is said to be *rotating*. In this case our task is first to show that there exists an additional Killing vector field, K^a , that is normal to H on H , besides t^a , and next, having the desired K^a , to find (a linear combination of) axisymmetric Killing fields $\varphi_{(i)}^a$ by $t^a = K^a + \Omega_{(i)}\varphi_{(i)}^a$, with $\Omega_{(i)}$ being some constants. Such a desired Killing vector field, K^a , is (1) first to be constructed locally in a neighborhood of H , and (2) then to be extended to the domain of outer communication. For the purpose of Step (2), we assume that the spacetime metric and matter fields be real analytic and then show that the ‘Taylor expansion’ at H satisfies

$$\underbrace{\mathcal{L}_\ell \mathcal{L}_\ell \cdots \mathcal{L}_\ell}_{m \text{ times}}(\mathcal{L}_K \Phi) = 0, \quad m = 0, 1, 2, \dots \quad \text{on } H, \quad (3.2)$$

where Φ collectively denotes all relevant physical fields and ℓ denotes some vector field that is transverse to H . Then, by analytic continuation we extend K^a to the entire spacetime.

In the following we focus on Step (1); we find a candidate K^a on the event horizon. The properties that the desired, candidate Killing field K^a should possess on H are: (i) $\mathcal{L}_t K^a = 0$ and $K^a K_a = 0$, (ii) $\mathcal{L}_K \Phi = 0$, and (iii) $K^c \nabla_c K^a = \alpha K^a$ with α being constant, which is to be identified with the surface gravity κ . Now let us choose a foliation of H by compact cross-sections Σ and decompose the stationary Killing vector t^a on H with respect to Σ as $t^a = n^a + s^a$, where n^a is null and s^a spacelike, tangent to Σ . (One can construct a well-behaved foliation by first choosing a single cross-section Σ_0 , and then Lie-dragging Σ_0 over H by the isometry of t^a .) Then, it is straightforward to check that n^a satisfies (i) and (ii). However, there is a prior no reason that α with respect to n^a needs be constant, since the decomposition $n^a = t^a - s^a$ depends upon the choice of Σ . (See figure 1.) Therefore our task is,

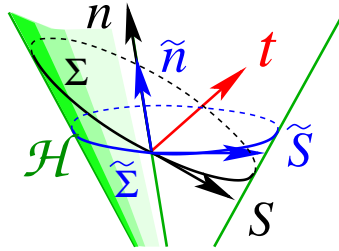


Fig. 1. The decomposition $t^a = n^a + s^a$ depends on the choice of foliation Σ . We wish to find a ‘correct’ foliation $\tilde{\Sigma}$ so that the corresponding $\tilde{n}^a = K^a$ satisfies—as a candidate Killing field—the desired properties (i)–(iii).

starting from an arbitrarily chosen Σ , to find the ‘desired’ foliation $\tilde{\Sigma}$ that gives rise

to $\tilde{n}^a = K^a$ with $\tilde{\alpha}$ being constant over H , satisfying the property (iii). It turns out that to find such a desired foliation $\tilde{\Sigma}$, one has to integrate along the orbit of s^a a set of two ordinary differential equations on Σ , both of which can be written in the form

$$\mathcal{L}_s \Psi = J, \quad (3.3)$$

where Ψ corresponds either to a function that correctly ‘normalizes’ n^a to obtain $\tilde{n}^a = K^a$ or to a coordinate function that defines the desired foliation, $\tilde{\Sigma}$, and where J is some smooth function on Σ . [See Lemma 2 of Ref. 50].] Now when solving this equation with respect to Ψ , the spacetime dimension and the topology of Σ play a crucial role. For 4-dimensions, the cross-section Σ must be topologically 2-sphere due to the topology theorem. It then immediately follows that the flow of s^a on Σ must have a fixed point, p , as the Euler characteristic of $\Sigma \approx S^2$ is non-zero. [See figure 2]. Now the (infinitesimal) action of s^a on any 2-vector, v^a , on the tangent

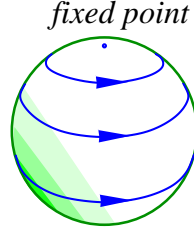


Fig. 2. A horizon cross-section Σ for a 4-dimensional black hole. The blue lines denote the flow of s^a , which has a fixed point and closed orbits on $\Sigma \approx S^2$.

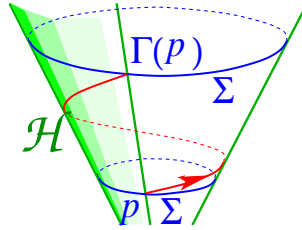


Fig. 3. An event horizon H of a 4-dimensional stationary black hole and a foliation of H by Σ . The red line denotes an orbit of t^a on H . A point p on a null geodesic generator of H is mapped by a discrete isometry Γ_t to a point $\Gamma_t(p)$ on the same null geodesic generator.

space at the fixed point p (where $s^b D_b v^a = 0$), is $\mathcal{L}_s v^a = -(D_b s^a) v^b$. Since $D_b s^a$ is a (2×2) anti-symmetric matrix, the action of s^a describes an infinitesimal ‘rotation’ on the tangent space at p . Therefore all the orbits of s^a must be closed with a certain period P . Then, by integrating Eq. (3.3) along a closed orbit of s^a one can always find a well-defined solution Ψ which gives rise to our desired foliation $\tilde{\Sigma}$ and the horizon normal Killing field $K^a = \tilde{n}^a$.

That the orbits of s^a are closed implies that t^a generates a discrete isometry, Γ_t , which maps each null generator of the event horizon into itself [see figure 3]. Therefore if we identify points in the spacetime that differ by the action of Γ_t with

period $t = P$, the event horizon becomes a compact null hypersurface ruled by closed null geodesic generators. Then, one can invoke Isenberg and Moncrief's symmetry theorem for compact Cauchy horizons,^{52), 53)} which provides the desired additional Killing field normal to the horizon as shown in Ref. 54). The discrete isometry Γ_t implies that the surface gravity is set to be

$$\kappa = \frac{1}{P} \int_0^P \alpha[\phi_s(x)] ds, \quad (3.4)$$

where $\phi_s(\cdot)$ denotes the flow generated by s^a and $x \in \Sigma$.

In higher dimensions $D > 4$, however, cross-sections Σ of the event horizon can admit non-trivial topology, and there is no reason that the isometries of Σ generated by s^a need have closed orbits even if s^a vanishes at some point $p \in \Sigma$. (This would be the case even in 4-dimensions if the horizon topology were non-spherical, e.g., torus). An example is supplied by considering a 5-dimensional Myers-Perry black hole solution,⁵⁵⁾ whose event horizon cross-section is topologically $\Sigma \approx S^3$. The solution admits two rotational Killing fields, φ_1^a , φ_2^a and their linear combination provides s^a on Σ . If we choose two rotational parameters in the linear combination so that their ratio becomes incommensurable, then the orbits of s^a do not have a closed orbit on Σ . Therefore, in general, there is no guarantee that one can find a well-defined solution, Ψ , for the higher dimensional black hole case. This

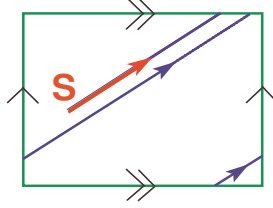


Fig. 4. The fundamental region of a 2-dimensional torus. Two-dimensional flat torus admits two Killing fields, each of which has closed orbits, but their linear combination s^a does not necessarily have closed orbits.

problem may be illustrated by a simple, lower-dimensional case; let us consider the case in which Σ is 2-dimensional flat torus and attempt to solve the same type of equation, (3.3), on Σ along a non-closed orbit of s^a on Σ . The torus Σ has two Killing fields $\tau_1^a = (\partial/\partial\tau_1)^a$ and $\tau_2^a = (\partial/\partial\tau_2)^a$, each of which has closed orbits on Σ . Then, s^a with non-closed orbits can be expressed as a linear combination

$$s^a = \Omega_1 \tau_1^a - \Omega_2 \tau_2^a, \quad (3.5)$$

with the ratio, Ω_1/Ω_2 , of the two coefficients being an irrational number. Then, it immediately follows that in terms of Fourier transform $\hat{J}(\mathbf{x}, m_1, m_2)$ of $J(\mathbf{x}, \tau_1, \tau_2)$, a formal solution Ψ to (3.3) is given by

$$\Psi(\mathbf{x}) = i \sum_{m_1, m_2} \frac{\hat{J}(\mathbf{x}, m_1, m_2)}{m_1 \Omega_1 - m_2 \Omega_2} = i \sum_{m_1, m_2} \frac{\hat{J}(\mathbf{x}, m_1, m_2)}{m_1 \Omega_2} \cdot \left| \frac{\Omega_1}{\Omega_2} - \frac{m_2}{m_1} \right|^{-1}. \quad (3.6)$$

Now recall that any irrational number, $\Omega_1/\Omega_2 \notin \mathbb{Q}$, can be approximated by some rational number $m_2/m_1 \in \mathbb{Q}$ as close as possible, by taking $m_1, m_2, \rightarrow \infty$ in an appropriate manner. This implies that the denominator of the right side of the above equation can become arbitrarily small and therefore that Ψ need not be convergent.

Nevertheless, this difficulty has been overcome for non-extremal black holes (i.e., the case in which $\kappa \neq 0$, see below) by employing a novel approach, and the rigidity result has been generalized to higher dimensions by Refs. 50), 51). We quote below the main theorems of Ref. 50) and briefly describe basic ideas of their proof, as well as the attempt of Ref. 56) to generalize these theorems to include extremal ($\kappa = 0$) black holes.

Theorem 3.1: [Reference 50)] *Let (M, g_{ab}) be an asymptotically flat, analytic stationary black hole solution to the vacuum Einstein equations. Assume further that the event horizon, H , of the black hole is analytic and is topologically $\mathbf{R} \times \Sigma$, with Σ compact and connected, and that $\kappa \neq 0$, where κ is given by Eq. (3.9) below. Then there exists a Killing field K^a , defined in a region that covers H and the entire domain of outer communication, such that K^a is normal to the horizon and K^a commutes with t^a .*

Theorem 3.2: [Reference 50)] *Under the same assumptions made in Theorem 3.1 above, if t^a is not tangent to the generators of H , then there exist mutually commuting Killing fields $\varphi_{(1)}^a, \dots, \varphi_{(j)}^a$ ($j \geq 1$) with closed orbits with period 2π which are defined in a region that covers H and the entire domain of outer communication. Each of these Killing fields commute with t^a , and*

$$t^a = K^a + \Omega_1 \varphi_{(1)}^a + \dots + \Omega_N \varphi_{(N)}^a, \quad (3.7)$$

for some constants Ω_i , all of whose ratios are irrational.

Theorem 3.2 implies that if the orbits of s^a fail to be closed, then the spacetime must admit at least two linearly independent rotational Killing fields.

A key new idea employed in Ref. 50) is to appeal to basic results of von Neumann ergodic theorem⁵⁷⁾—which relies only on the compactness of Σ , and yields that there exists a long-time average:

$$\alpha^*(x) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \alpha[\phi_s(x)] ds. \quad (3.8)$$

Then, using the vacuum Einstein equations, one can show that the limit α^* is constant and indeed coincides with the spatial average,

$$\kappa := \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} \alpha(x) d\Sigma. \quad (3.9)$$

Furthermore, when $\kappa \neq 0$, one can find from the Einstein equations, a differential relation $D_a J = \mathcal{L}_s \beta_a$ for each of the two (3.3), with β_a being some real analytic vector field on Σ related to a metric component. Then, one can find Ψ as a real analytic solution of $D^a D_a \Psi = D_a \beta_a$ on Σ . The ergodic theorem also helps to fix

the freedom of adding an solution of the homogeneous equations, $D^a D_a \Psi = 0$ and $\mathcal{L}_s \Psi = 0$.

For the extremal black hole case $\kappa = 0$, however, the above method using the ergodic theory does not seem to apply, since, for $\kappa = 0$, we do not appear to have a differential relation analogous to $D_a J = \mathcal{L}_s \beta_a$ for one of the two (3.3). To proceed, note that in the example of 2-torus above, if for some $q > 0$,

$$\left| \frac{\Omega_1}{\Omega_2} - \frac{m_2}{m_1} \right| > \frac{1}{m_1^q},$$

then one can show that the formal solution, Ψ , to (3.3), given by Eq. (3.6), becomes convergent and well-defined. This condition—called the *Diophantine condition*—does *not* hold when Ω_1/Ω_2 is a Liouville number. However, such a number is known to be in a set of *measure zero*. Therefore we can virtually always solve Eq. (3.3).

For the problem of how to solve Eq. (3.3), we note that due to the fact that s^a generates an isometry group on a compact Riemannian manifold Σ , it can locally be decomposed in terms of N (≥ 1) Killing fields φ_i^a ($i = 1, \dots, N$) as

$$s^a = \Omega_1 \varphi_1^a + \dots + \varphi_N^a, \quad (3.10)$$

with $\Omega_i \in \mathbf{R}^N$. Then the following lemma has been shown by Ref. 56).

Lemma 3.3: [Reference 56)] *Let J be a smooth function on Σ with the property*

$$0 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T J \circ \phi_s ds. \quad (3.11)$$

Let $\underline{\Omega} = (\Omega_1, \dots, \Omega_N) \in \mathbf{R}^N$ satisfy the following “Diophantine condition”: There exists a number q such that

$$|\underline{\Omega} \cdot \underline{m}| > |\underline{\Omega}| \cdot |\underline{m}|^{-q} \quad (3.12)$$

holds for all but finitely many $\underline{m} \in \mathbf{Z}^N$. Then the equation (3.3) with s^a as in Eq. (3.10), has a smooth solution Ψ on Σ . Furthermore, if J is real analytic, then the same statements hold true and Ψ is real analytic.

With the additional conditions of (3.12), the rigidity theorems above have been extended to include extremal black holes in Theorems 1 and 2 of Ref. 56). Note that when $N = 1$, the Diophantine condition is automatically satisfied, and when $N > 1$ —which can happen only in higher dimensions—the condition is non-trivial. In this sense, the theorems for the extremal black hole case are weaker than the theorems for the non-extremal case. However, one also should note that the Diophantine condition holds for all $\underline{\Omega} \in \mathbf{R}^N$ except for a set of Lebesgue measure zero.

A few remarks on the rigidity theorems for both extremal and non-extremal cases are in order: Theorems 3.1 and 3.2 above, and those corresponding to the extremal case in Ref. 56) hold also true for stationary black holes coupled to matter fields in a fairly general class of theories that include multiple of scalar fields with arbitrary potentials, Abelian gauge fields, as well as cosmological constant. Thus,

the above theorems in particular apply to stationary, asymptotically anti-de Sitter black holes as well.

The theorems apply not only to a black hole horizon but also to any horizon defined as a “boundary” of the causal past of a complete orbit of some Killing vector field, such as a cosmological horizon if exists.

One can partially remove the analyticity assumption for the black hole interior, following the strategy of Refs. 54), 58). For the non-extremal case, the event horizon is isometric to a portion of some bifurcate Killing horizon.^{59), 60)} Then one can use the bifurcate horizon as an initial data surface for K^a defined in a neighborhood of H . Then, applying a characteristic initial value formulation to extend K^a into the interior of the black hole. This type of characteristic initial value problem is ill-defined toward the black hole exterior region and therefore would not appear to be applicable to extend K^a into the domain of outer communication. Nevertheless, a remarkable progress has recently been made along this direction.^{61), 62)}

The staticity theorem of Sudarsky and Wald³³⁾ can be generalized straightforwardly to higher dimensions. Then, combined with the rigidity theorems, it implies that a stationary, non-extremal black hole in $D \geq 4$ Einstein-Maxwell system must be either static or axisymmetric.⁶³⁾ For the former case, we can bypass Steps (iii) and (iv) of the proof when generalizing the uniqueness theorem to higher dimensions as in the next section.

§4. Uniqueness of static black holes

In this section, we will discuss the uniqueness of static black holes in asymptotically flat spacetimes. As seen in chapter 1 and 4 there seems to be a large variety of exact solutions with different horizon topology, and one cannot restrict the shape of the horizon to be spherical in general stationary spacetimes before showing the uniqueness. However, we can show the uniqueness of the Schwarzschild-Tangherlini spacetimes⁶⁴⁾ in vacuum static spacetimes, and it is shown that the topology is sphere at the same time.^{47), 65), 48)}

In 4 dimensions, there are two ways to prove the uniqueness: one by Israel³⁴⁾ and the other by Bunting & Masood-ul-Alam.⁶⁶⁾ However, both of the methods rely on the special nature of 4-dimensional spacetimes. In the former method,³⁴⁾ the Gauss-Bonnet theorem, that is, $\int_{\Sigma} {}^{(2)}\mathcal{R} = 8\pi$, is used, where Σ is a two-dimensional compact surface and ${}^{(2)}\mathcal{R}$ is the Ricci scalar. In the latter method,⁶⁶⁾ one uses the Weyl-Bach tensor, which vanishes if and only if the three dimensional space is conformally flat. Therefore, we need a new way to avoid using these four-dimensional specialities.

We first describe some basic formulae for the proof and look at the outline of the proof. Then we show the proof of the uniqueness theorem for vacuum^{47), 65)} and electro-vacuum cases.⁴⁸⁾

4.1. Basic tools

In this subsection, we describe the basics which will be used for the proof of the static black hole uniqueness. The formulae here will hold in general static space-

times. This is because we will not use the field equations. As commented above, non-rotating black hole spacetimes are shown to be static.^{33),63)} By definition, the staticity means the existence of a timelike Killing vector $\xi = \partial/\partial t$ which is hypersurface orthogonal. Therefore, we can write down the metric as

$$ds^2 = -V^2(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j, \quad (4.1)$$

where $\{x^i\}$ are the spatial coordinates. Note that the components of the metric do not depend on the time coordinate $x^0 = t$ and $g_{0i} = 0$. We denote $t = \text{constant}$ surface by S . In this situation, it is natural to consider $(D-1) + 1$ decomposition of the geometrical quantities. If black holes exist, they are located at $V = 0$, that is, the Killing horizon.

The D -dimensional Riemann tensor R_{MNKL} is decomposed into

$$R_{ijkl} = {}^{(D-1)}R_{ijkl} \quad (4.2)$$

and

$$R_{0i0j} = V D_i D_j V, \quad (4.3)$$

where ${}^{(D-1)}R_{ijkl}$ and D_i are the $(D-1)$ -dimensional Riemann tensor and the covariant derivative with respect to g_{ij} , respectively. The D -dimensional Ricci tensor is decomposed into

$$R_{00} = V D_i D^i V \quad (4.4)$$

and

$$R_{ij} = {}^{(D-1)}R_{ij} - \frac{1}{V} D_i D_j V, \quad (4.5)$$

where ${}^{(D-1)}R_{ij}$ is the $(D-1)$ -dimensional Ricci tensor.

The asymptotically flat conditions are

$$V = 1 - \frac{m}{r^{D-3}} + O(1/r^{D-2}) \quad (4.6)$$

$$g_{ij} = \left(1 + \frac{2}{D-3} \frac{m}{r^{D-3}}\right) \delta_{ij} + O(1/r^{D-2}), \quad (4.7)$$

where $r := \sqrt{\delta_{ij} x^i x^j}$ and m is proportional to the ADM mass by

$$m = \frac{8\pi G}{(D-2)\text{Vol}(S^{D-2})} M. \quad (4.8)$$

We can check them by the linear perturbation of the metric.

Since V often behaves like a monotonic increasing function for the outward direction, we may employ V as a radial coordinate. So we want to introduce the unit normal vector as

$$n_i = \rho D_i V, \quad (4.9)$$

where ρ is the “lapse” function defined by $\rho := (D_i V D^i V)^{-1/2}$. Then one defines the induced metric orthogonal to $\xi = \partial/\partial t$ and n_i as

$$h_{ij} = g_{ij} - n_i n_j. \quad (4.10)$$

The extrinsic curvature of $V = \text{constant}$ surfaces is defined as

$$k_{ij} = h_i^k D_k n_j, \quad (4.11)$$

where $h_i^j = g^{jk} h_{ik}$.

To examine the regularity of spacetimes, we often look at the Kretschmann invariant

$$\begin{aligned} R_{IJKL} R^{IJKL} &= R_{ijkl} R^{ijkl} + 4R_{0i0j} R^{0i0j} \\ &= {}^{(D-1)}R_{ijkl} {}^{(D-1)}R^{ijkl} + 4V^{-2} D_i D_j V D^i D^j V, \end{aligned} \quad (4.12)$$

where we used Eq. (4.3) in the second line. We will rewrite the second term in the second line. To do so we go back to the definition of the unit vector in the following form

$$D_j V = \rho^{-1} n_j. \quad (4.13)$$

Operating the derivative D_i to the above equation, we have

$$D_i D_j V = \frac{1}{\rho} k_{ij} - \frac{1}{\rho^2} (\mathcal{D}_i \rho n_j + \mathcal{D}_j \rho n_i) - \frac{1}{\rho^2} n_i n_j n^k D_k \rho, \quad (4.14)$$

where \mathcal{D}_i is the covariant derivative with respect to h_{ij} . Taking the trace of the above equation, we also have the formula

$$k = \rho D_i D^i V + \frac{1}{\rho} n^i D_i \rho. \quad (4.15)$$

Using the above formulae, the Kretschmann invariant is now rewritten as

$$\begin{aligned} R_{IJKL} R^{IJKL} &= {}^{(D-1)}R_{ijkl} {}^{(D-1)}R^{ijkl} \\ &\quad + \frac{4}{V^2} \left[\frac{1}{\rho^2} k_{ij} k^{ij} + \frac{1}{\rho^4} (n^i D_i \rho)^2 + \frac{2}{\rho^4} (\mathcal{D} \rho)^2 \right]. \end{aligned} \quad (4.16)$$

In general, the regularity condition on the horizon ($V = 0$) implies

$$k_{ij}|_{V=0} = n^i D_i \rho|_{V=0} = \mathcal{D}_i \rho|_{V=0} = 0. \quad (4.17)$$

From Eq. (4.15) we see that

$$D_i D^i V|_{V=0} = 0 \quad (4.18)$$

holds. Together with the Einstein equation, we will also have the constraints on matter fields at the horizon.

Finally we consider the conformal transformation

$$\tilde{g}_{ij} = \Omega^2 g_{ij}. \quad (4.19)$$

The Ricci scalar is transformed as

$$\Omega^{2(D-1)}\tilde{R} = {}^{(D-1)}R - 2(D-2)D_i D^i \ln \Omega - (D-3)(D-2)(D_i \ln \Omega)^2, \quad (4.20)$$

where $(D_i \ln \Omega)^2 = D_i \ln \Omega D^i \ln \Omega$. The unit normal vector and induced metric are transformed as

$$\tilde{n}_i = \Omega n_i \quad (4.21)$$

and

$$\tilde{h}_{ij} = \tilde{g}_{ij} - \tilde{n}_i \tilde{n}_j = \Omega^2(g_{ij} - n_i n_j) = \Omega^2 h_{ij}. \quad (4.22)$$

For the extrinsic curvature, we see

$$\tilde{k}_{ij} = \tilde{h}_i^k \tilde{D}_k \tilde{n}_j = \Omega k_{ij} + h_{ij} n^k D_k \Omega. \quad (4.23)$$

In the proof of the uniqueness, we will use the following version of the positive mass theorem.

Theorem 4.1: [Reference 68), 67)] *Let us consider $(D-1)$ -dimensional asymptotically flat slices with the non-negative Ricci scalar. Then the ADM mass is non-negative. Moreover, the slice is flat if and only if the ADM mass vanishes.*

Remark: In the above theorem, it is assumed that the slice does not have boundaries except for the infinities, and that the metric is C^1 . Although the assumptions for the positive mass theorem are going to be relaxed due to various efforts, we would restrict ourself to minimal consideration. Because we suppose that the space-time manifold is spin (it is defined so that spinors satisfying a Dirac-type equation exist), the positive mass theorem holds for any dimensions ($D \geq 4$).

4.2. Outline of proof

There are several different black hole solutions, possessing different types of hairs, and detail of the uniqueness proof depends on what type of hairs black hole solutions of interest have. It is possible to show the uniqueness for a fairly general class of static black holes that possess many different hairs. However, for such a general case, the proof gets complicated and it is hard to see the heart of the proof. For this reason, we will first describe the common part of the proof and then go into details for a couple of specific cases.

One often assumes that the outside of black holes is vacuum. Once one puts some matter fields or cosmological constant, it becomes difficult to show the uniqueness. We also discuss this point shortly.

Let us look at the outline of the proof. In the proof, we first show that the static slice is conformally flat, that is, there is a conformal transformation $\tilde{g}_{ij} = \Omega^2 g_{ij}$ so that the resultant space is flat ($\tilde{g}_{ij} = \delta_{ij}$). The conformal flatness is shown by the positive mass theorem. More precisely, we can show the non-negativity of the Ricci scalar of the conformally transformed space and the vanishing of the ADM mass.

The positive mass theorem implies that the space with non-negative Ricci scalar and zero mass must be flat. As a result, we realize that the static slice is conformally flat. Then we can find a harmonic function, say u , (so $\Delta u = 0$) constructed from the conformal factor Ω or a part on the flat space and the boundary associated with the horizons is spherically symmetric. The problem is now equivalent with the electrostatic potential with the spherical boundary and we know that each level surface of the potential (each $u = \text{constant}$ surfaces) must be spherically symmetric. Finally we can show that the original space (S, g_{ij}) is spherically symmetric. The static and spherically symmetric spacetimes will be a known exact solution from the direct calculation or the Birkhoff theorem. For the vacuum case, the spacetime is the Schwarzschild-Tangherlini one. This is the outline of the proof.

The main task in the proof is to find an appropriate conformal transformation for each theory so that the positive mass theorem can be applied. When we know the exact solution, we can guess the conformal transformation from the solution itself.

4.3. Vacuum case

Let us first consider the vacuum case. This is the simplest case. The vacuum Einstein equation is

$$R_{MN} = 0. \quad (4.24)$$

The black hole solution which we know is the Schwarzschild-Tangherlini solution

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{D-2}^2, \quad (4.25)$$

where

$$f(r) = 1 - \frac{2m}{r^{D-3}}, \quad (4.26)$$

where m is proportional to the ADM mass as Eq. (4.8). In the vacuum case, $R_{00} = 0$ gives us

$$D_i D^i V = 0. \quad (4.27)$$

Then V is a harmonic function in $t = \text{constant}$ hypersurfaces (S, g_{ij}) . Since this harmonic function does not have the maximum, we can employ V as a radial coordinate on S . We will discuss the geometrical feature of $V = \text{constant}$ surfaces (the level surfaces of V).

We employ the following two conformal transformations

$$\tilde{g}_{ij}^\pm = \Omega_\pm^2 g_{ij}, \quad (4.28)$$

where

$$\Omega_\pm = \left(\frac{1 \pm V}{2} \right)^{\frac{2}{D-3}} =: \omega_\pm^{\frac{2}{D-3}}. \quad (4.29)$$

Then, from the asymptotic conditions (Eqs. (4.6) and (4.7)), we can see the asymptotic behavior of \tilde{g}_{ij}^\pm at “infinity” ($r = \infty$)

$$\begin{aligned} \tilde{g}_{ij}^+ &= \left(1 - \frac{m}{2r^{D-3}} \right)^{\frac{4}{D-3}} \left(1 + \frac{2}{D-3} \frac{m}{r^{D-3}} \right) \delta_{ij} + O(1/r^{D-2}) \\ &= \delta_{ij} + O(1/r^{D-2}) \end{aligned} \quad (4.30)$$

and

$$\begin{aligned}
\tilde{g}_{ij}^- &\simeq \left(\frac{m}{2r^{D-3}} \right)^{\frac{4}{D-3}} (dr^2 + r^2 d\Omega_{D-2}^2) \\
&= (m/2)^{4/(D-3)} r^{-4} (dr^2 + r^2 d\Omega_{D-2}^2) \\
&= dR^2 + R^2 d\Omega_{D-2}^2,
\end{aligned} \tag{4.31}$$

where $R := (m/2)^{2/(D-3)} r^{-1}$. The reason why we can consider two conformal transformation will be seen soon later. The mass vanishes in $(\tilde{S}_+, \tilde{g}_{ij}^+)$. The infinity $r = \infty$ in (S, g_{ij}) corresponds to be a point in $(\tilde{S}_-, \tilde{g}_{ij}^-)$. The Ricci scalar of \tilde{g}_{ij}^\pm becomes

$$(D-1)\tilde{R}_\pm = 0. \tag{4.32}$$

Let us try to construct the manifold $(\tilde{S}, \tilde{g}_{ij}) := (\tilde{S}_+, \tilde{g}_{ij}^+) \cup (\tilde{S}_-, \tilde{g}_{ij}^-) \cup \{p\}$ gluing along the $V = 0$ surfaces and adding the point p . $(\tilde{S}_-, \tilde{g}_{ij}^-) \cup \{p\}$ buries the “hole” of $(\tilde{S}_+, \tilde{g}_{ij}^+)$. On the horizon, it is easy to see that the induced metric in $(\tilde{S}_\pm, \tilde{g}_{ij}^\pm)$ is continuous, that is,

$$\tilde{h}_{ij}^+|_{V=0} = \tilde{h}_{ij}^-|_{V=0}. \tag{4.33}$$

The extrinsic curvature of the $V = 0$ surface in $(\tilde{S}_\pm, \tilde{g}_{ij}^\pm)$ has the same absolute value with opposite signature

$$\tilde{k}_{ij}^\pm|_{V=0} = \pm \frac{2^{\frac{D-1}{D-3}}}{D-3} \frac{1}{\rho_0} \tilde{h}_{ij}^\pm|_{V=0}. \tag{4.34}$$

The surface satisfying the relation (4.34) is said to be totally umbilical. The difference of the signature comes from that of the normal direction to the $V = 0$ surfaces. Since the extrinsic curvature is expressed by the Lie derivative as

$$\tilde{k}_{ij} = \frac{1}{2} \mathcal{L}_{\tilde{n}} \tilde{h}_{ij}, \tag{4.35}$$

the metric of \tilde{g}_{ij} is C^1 . And \tilde{S} does not have the boundary except for the infinities.

Then we can apply the positive mass theorem to $(\tilde{S}, \tilde{g}_{ij})$ and show that $(\tilde{S}, \tilde{g}_{ij})$ is flat,

$$\tilde{g}_{ij}^\pm = \delta_{ij}. \tag{4.36}$$

We also note that the totally umbilical surfaces in the flat space is spherically symmetric.⁶⁹⁾

The equation for V , $D^2V = 0$, is rewritten in terms of g_{ij} as follows

$$\tilde{D}^i [\Omega_\pm^{-(D-3)} \tilde{D}_i V] = 0. \tag{4.37}$$

Then, defining a function u_\pm by

$$u_\pm = \frac{1}{\Omega_\pm} = \frac{2}{1 \pm V}, \tag{4.38}$$

we see that

$$\tilde{D}_i \tilde{D}^i u_{\pm} = \Delta_0 u_{\pm} = 0 \quad (4.39)$$

holds, where Δ_0 is the Laplacian of the flat space.

Now the problem here can be reduced to the electrostatic potential with the spherical boundary in the flat space. Then it is easy to see that the $u_{\pm}=\text{constant}$ surfaces are spherically symmetric in $(\tilde{S}_+, \tilde{g}_{ij}^+)$. Since u_{\pm} and Ω_{\pm} are the functions of V only, $V=\text{constant}$ surfaces are also spherically symmetric in (S, g_{ij}) . Through the usual computation, it is shown that a static, spherically symmetric vacuum black hole solution is isometric to the Schwarzschild-Tangherlini metric. This is the end of the proof.

Before closing this subsection, we will express the spacetimes in the isotropic coordinates to guess the conformal transformation for the proof of the uniqueness. This may give us a hint for other black holes. In the isotropic coordinates, the metric of the Schwarzschild-Tangherlini spacetime is

$$ds^2 = -\frac{\left(1 - \frac{m}{2\rho^{D-3}}\right)^2}{\left(1 + \frac{m}{2\rho^{D-3}}\right)^2} dt^2 + \left(1 + \frac{m}{2\rho^{D-3}}\right)^{\frac{4}{D-3}} (d\rho^2 + \rho^2 d\Omega_{D-2}^2), \quad (4.40)$$

where the relation between r and ρ is given by

$$r = \rho \left(1 + \frac{m}{2\rho^{D-3}}\right)^{\frac{2}{D-3}}. \quad (4.41)$$

Compared to the general form of the static spacetime (Eq. (4.1)), we can identify V as

$$V = \frac{1 - \frac{m}{2\rho^{D-3}}}{1 + \frac{m}{2\rho^{D-3}}}. \quad (4.42)$$

Then one of the conformal transformations is expected to be

$$\Omega_+ = \left(1 + \frac{m}{2\rho^{D-3}}\right)^{-\frac{2}{D-3}} = \left(\frac{1+V}{2}\right)^{\frac{2}{D-3}}. \quad (4.43)$$

Introducing the new coordinate defined by

$$R := \rho^{-1} (m/2)^{\frac{2}{D-3}}, \quad (4.44)$$

the spatial part of the metric is rewritten as

$$d\ell^2 = \left(1 + \frac{m}{2R^{D-3}}\right)^{\frac{4}{D-3}} (dR^2 + R^2 d\Omega_{D-2}^2). \quad (4.45)$$

In this coordinate, $\rho = \infty$ corresponds to $R = 0$. Then we can expect the other conformal transformation to be

$$\Omega_- = \left(1 + \frac{m}{2R^{D-3}}\right)^{-\frac{2}{D-3}} = \left[\frac{\rho^{D-3}}{m/2} \left(1 + \frac{m}{2\rho^{D-3}}\right)\right]^{-\frac{2}{D-3}} = \left(\frac{1-V}{2}\right)^{\frac{2}{D-3}}. \quad (4.46)$$

They are exactly the same as those employed in the proof of the uniqueness theorem.

4.4. *Electro-vacuum case*

Next we will consider charged black hole solutions. The Lagrangian is given by

$$16\pi GL = R - F_{MN}F^{MN}, \quad (4.47)$$

where F_{MN} is the field strength of the Maxwell fields. The Einstein equation and Maxwell equation are

$$R_{MN} = 2F_M^K F_{NK} - \frac{1}{D-2}g_{MN}F^2 \quad (4.48)$$

and

$$\nabla_M F^{MN} = 0 \quad (4.49)$$

with $\nabla_{[M}F_{NI]} = 0$. In general static spacetimes, the electric part of the Maxwell field becomes

$$F_{0i} = -\partial_i\psi(x^j). \quad (4.50)$$

Here we note that one may be interested in the higher form fields because they are essential ingredient in string theory. However, the possible global charge is only for the Maxwell field. This is because to compute the global charge, we need to perform an integration of the field strength over a sphere with infinite radius. To specify the sphere in spacetimes, we need two directions which are orthogonal to the sphere. Therefore, we cannot define the global charge of the higher form field strength. We will comment on the hair of higher form fields in the final subsection again.

The solution which we know is the Reissner-Nordström solution

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_{D-2}^2, \quad (4.51)$$

where

$$f(r) = 1 - \frac{2m}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}}. \quad (4.52)$$

The electric potential ψ is given by

$$\psi = \frac{Q/C}{r^{D-3}}, \quad (4.53)$$

where

$$C := \left(\frac{2(D-3)}{D-2}\right)^{1/2}. \quad (4.54)$$

We shall show the uniqueness of the Reissner-Nordström solution in the Einstein-Maxwell theory. In this subsection we assume $m > |Q|$. We will briefly comment on the extreme case of $m = |Q|$ in the final subsection.

From Eq. (4.48), we have

$$R_{00} = V D_i D^i V = C^2 (D_i \psi)^2, \quad (4.55)$$

and

$$R_{ij} = {}^{(D-1)}R_{ij} - \frac{1}{V}D_iD_jV = -\frac{2}{V^2}D_i\psi D_j\psi + \frac{2}{D-2}g_{ij}\frac{1}{V^2}(D_k\psi)^2. \quad (4.56)$$

The Maxwell equation becomes

$$D_iD^i\psi = \frac{D^iV D_i\psi}{V}. \quad (4.57)$$

The asymptotic condition for ψ is given by

$$\psi = \frac{Q/C}{r^{D-3}} + O(1/r^{D-2}). \quad (4.58)$$

Let us consider the following conformal transformation

$$\tilde{g}_{ij} = \Omega_{\pm}^2 g_{ij}, \quad (4.59)$$

$$\Omega_{\pm} = \left[\left(\frac{1 \pm V}{2} \right)^2 - \frac{C^2}{4}\psi^2 \right]^{\frac{1}{D-3}} =: \omega_{\pm}^{\frac{1}{D-3}} = (\lambda_{\pm}\mu_{\pm})^{\frac{1}{D-3}}, \quad (4.60)$$

where

$$\lambda_{\pm} = \frac{1 \pm V}{2} - \frac{C}{2}\psi \quad (4.61)$$

and

$$\mu_{\pm} = \frac{1 \pm V}{2} + \frac{C}{2}\psi. \quad (4.62)$$

From the later discussion about the exact solution, we will see why they are chosen. The asymptotic behavior of \tilde{g}_{ij}^{\pm} is

$$\tilde{g}_{ij}^{+} = \delta_{ij} + O(1/r^{D-2}) \quad (4.63)$$

and

$$\tilde{g}_{ij}^{-} \simeq dR^2 + R^2 d\Omega_{D-2}^2, \quad (4.64)$$

where

$$r^{-1} = \left(\frac{m^2 - Q^2}{4} \right)^{\frac{1}{D-3}} R. \quad (4.65)$$

As in the vacuum case, the ADM mass in $(\tilde{S}_+, \tilde{g}_{ij}^{+})$ vanishes and the infinity ($r = \infty$) becomes to be a point p in $(\tilde{S}_-, \tilde{g}_{ij}^{-})$.

From the Einstein equation, the $(D-1)$ -dimensional Ricci scalar is computed as

$${}^{(D-1)}R = 2 \frac{(D_i\psi)^2}{V^2}. \quad (4.66)$$

Then the Ricci scalar of \tilde{g}_{ij}^\pm becomes

$$\begin{aligned} \Omega_\pm^{2(D-1)} \tilde{R}_\pm &= 2 \frac{(D_i \psi)^2}{V^2} - \frac{2(D-2)}{(D-3)} \left(\frac{D_i D^i \lambda_\pm}{\lambda_\pm} + \frac{D_i D^i \mu_\pm}{\mu_\pm} \right) \\ &\quad + \frac{D-2}{D-3} \left(\frac{D_i \lambda_\pm}{\lambda_\pm} - \frac{D_i \mu_\pm}{\mu_\pm} \right)^2. \end{aligned} \quad (4.67)$$

Here we note that, for λ_\pm and μ_\pm , the following equations hold

$$D_i D^i \lambda_\pm = \mp \frac{C}{V} D^i \psi D_i \lambda_\pm \quad (4.68)$$

$$D_i D^i \mu_\pm = \pm \frac{C}{V} D^i \psi D_i \mu_\pm. \quad (4.69)$$

Using them, the Ricci scalar of \tilde{g}_{ij}^\pm can be expressed as

$$\begin{aligned} \Omega_\pm^{2(D-1)} \tilde{R}_\pm &= \frac{2}{C^2} \left(\frac{D_i \lambda_\pm}{\lambda_\pm} - \frac{D_i \mu_\pm}{\mu_\pm} \pm C \frac{D_i \psi}{V} \right)^2 \\ &= \frac{C^2}{8\omega_\pm^2 V^2} \left[2\psi V D_i V + (1 - V^2 - C^2 \psi^2) D_i \psi \right]^2. \end{aligned} \quad (4.70)$$

Then we can see

$$^{(D-1)}\tilde{R}_\pm \geq 0. \quad (4.71)$$

The regularity on the horizon implies

$$k_{ij}|_{V=0} = \mathcal{D}_i \rho|_{V=0} = D_i \psi|_{V=0} = 0. \quad (4.72)$$

Then the induced metric and the extrinsic curvature of $V = 0$ surface in $(\tilde{S}, \tilde{g}_{ij})$ become

$$\tilde{h}_{ij}^+|_{V=0} = \tilde{h}_{ij}^-|_{V=0} \quad (4.73)$$

$$\tilde{k}_{ij}|_{V=0} = \pm \frac{2^{\frac{D-1}{D-3}}}{(D-3)(1 - C^2 \psi_0^2)^{\frac{D-2}{D-3}}} \frac{1}{\rho_0} \tilde{h}_{ij}^\pm|_{V=0}. \quad (4.74)$$

Let us construct the manifold $(\tilde{S}, \tilde{g}_{ij}) := (\tilde{S}_+, \tilde{g}_{ij}^+) \cup (\tilde{S}_-, \tilde{g}_{ij}^-) \cup \{p\}$. As in the vacuum case, the metric and extrinsic curvature are continuous. Thus we can apply the positive mass theorem to $(\tilde{S}, \tilde{g}_{ij})$. Since $(\tilde{S}, \tilde{g}_{ij})$ is massless and the Ricci scalar is non-negative, the positive mass theorem implies that $(\tilde{S}, \tilde{g}_{ij})$ must be flat, that is,

$$\tilde{g}_{ij} = \delta_{ij}. \quad (4.75)$$

The expression of $^{(D-1)}\tilde{R}_\pm$ gives us

$$2V\psi D_i V + (1 - V^2 - C^2 \psi^2) D_i \psi = 0. \quad (4.76)$$

This means that the level surface of V is equivalent with that of ψ .

Now we can realize that Eqs. (4.68) and (4.69) are rewritten in terms of \tilde{g}_{ij} as

$$\lambda_{\pm}^2 \tilde{D}^2 \lambda_{\pm}^{-1} = -\tilde{D}^i \lambda_{\pm} \left(\tilde{D}_i \ln \frac{\mu_{\pm}}{\lambda_{\pm}} \mp \frac{C}{V} \tilde{D}_i \psi \right) = 0 \quad (4.77)$$

and

$$\mu_{\pm}^2 \tilde{D}^2 \mu_{\pm}^{-1} = \tilde{D}^i \mu_{\pm} \left(\tilde{D}_i \ln \frac{\mu_{\pm}}{\lambda_{\pm}} \mp \frac{C}{V} \tilde{D}_i \psi \right) = 0. \quad (4.78)$$

Therefore, λ_{\pm}^{-1} and μ_{\pm}^{-1} are harmonic functions in $(\tilde{S}_{\pm}, \tilde{g}_{ij}^{\pm})$. Each level surface is the same. Then we take λ_{+}^{-1} to show the spherical symmetry. As in the vacuum case, $V = 0$ surfaces in $(\tilde{S}, \tilde{g}_{ij})$ is totally umbilical and this means that the $V = 0$ surface is spherically symmetric in the flat space. Thus, the problem is reduced to be the electro-static potential one with spherically symmetric boundary. So the level surfaces of λ_{+}^{-1} are all spherically symmetric in $(\tilde{S}_{+}, \tilde{g}_{ij}^{+})$. Since $\Omega_{+} = \Omega_{+}(V, \psi)$ and the level surfaces of V are the same as those of ψ , $g_{ij} = \Omega_{+}^{-2} \delta_{ij}$ is also spherically symmetric. Solving the Einstein-Maxwell equations directly, we can see that the solution is isometric to the Reissner-Nordström metric. This is the end of the proof.

At first glance the choice of the conformal transformation is non-trivial although it is quite important. The choice can be understood from the Reissner-Nordström solution in the isotropic coordinates

$$ds^2 = - \left(\frac{1 - \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}}{1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}} \right)^2 dt^2 + \left(1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}} \right)^{\frac{2}{D-3}} (d\rho^2 + \rho^2 d\Omega_{D-2}^2), \quad (4.79)$$

where ρ is introduced through the relation of

$$r = \rho \left(1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}} \right)^{\frac{1}{D-3}}. \quad (4.80)$$

The electric potential is

$$\psi = \frac{Q}{\rho^{D-3}} \frac{1}{1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}}. \quad (4.81)$$

Compared to the metric form of general static spacetimes, we can identify V as

$$V = \frac{1 - \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}}{1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}}. \quad (4.82)$$

Since

$$\left(\frac{1+V}{2} \right)^2 - \frac{C^2}{4} \psi^2 = \frac{1}{1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}}, \quad (4.83)$$

we may choose the conformal transformation

$$\begin{aligned}\Omega_+ &= \left(1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}\right)^{-\frac{1}{D-3}} \\ &= \left[\left(\frac{1+V}{2}\right)^2 - \frac{C^2}{4}\psi^2\right]^{\frac{1}{D-3}}.\end{aligned}\quad (4.84)$$

Introducing the new coordinate

$$R := \left(\frac{m^2 - Q^2}{4}\right)\rho^{-1}, \quad (4.85)$$

the spatial part of the metric is written as

$$d\ell^2 = \frac{\rho^4}{\left(\frac{m^2 - Q^2}{4}\right)^{\frac{2}{D-3}}} \left(1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}\right)^{\frac{2}{D-3}} (dR^2 + R^2 d\Omega_{D-2}^2) \quad (4.86)$$

Note that

$$\left(\frac{1-V}{2}\right)^2 - \frac{C^2}{4}\psi^2 = \frac{\frac{m^2 - Q^2}{4}}{\rho^{2(D-3)}} \frac{1}{1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}} \quad (4.87)$$

holds. They also give us a hint for another choice of the conformal transformation, that is,

$$\begin{aligned}\Omega_- &= \left[\frac{\rho^{2(D-3)}}{\frac{m^2 - Q^2}{4}} \left(1 + \frac{m}{\rho^{D-3}} + \frac{m^2 - Q^2}{4} \frac{1}{\rho^{2(D-3)}}\right)\right]^{-\frac{1}{D-3}} \\ &= \left[\left(\frac{1-V}{2}\right)^2 - \frac{C^2}{4}\psi^2\right]^{\frac{1}{D-3}}.\end{aligned}\quad (4.88)$$

4.5. Other cases

From the view point of simpleness, we have considered the vacuum and electrovacuum cases. There are also uniqueness theorems for other static black holes. We will briefly comment on each case.

(i) *Dilatonic black holes*: In higher dimensions, we have the exact solution of the dilatonic Einstein-Maxwell theory.⁷⁰⁾ In this case, we can also prove the uniqueness in the same manner here.⁴⁸⁾ The same technique also applies to the case coupled to sigma-model fields.⁷¹⁾

(ii) *No-dipole hair theorem for form fields*: One may be interested in the hair of the p -form field strength of black holes. When one can show the no-hair, it is, of course, expected that exact solutions with such hair cannot exist. Therefore, we cannot have the hint for the conformal transformation from solutions because there are no known hairy solutions. For $p \geq 3$, possible hairs are dipole or higher multipole components of the field strengths. However, we can show the no-hair of the electric p -form field

strength when $p \geq (D+1)/2$.⁷²⁾ In the proof of this theorem, we use the same conformal transformation employed in the proof of the Schwarzschild-Tangherlini spacetime.

(iii) *Extreme black holes in Einstein-Maxwell theory*: In the previous subsection, the non-extreme condition $m > |Q|$ is imposed. In the extreme limit, there are exact solutions for multi-black holes. The uniqueness of such solutions has been proven.^{73),74)} See Ref. 75) for some related work.

(iv) *Asymptotically de Sitter/anti-de Sitter*: There are exact solutions of the Schwarzschild-(anti-)de Sitter solutions. See Refs. 76), 77) for attempts to generalize the static uniqueness to the asymptotically anti-de Sitter case. See also Ref. 78) for perturbative approach.

There are also asymptotically flat supersymmetric black hole solutions in four and five dimensions. Since they are extreme type, they will be multi-black hole solutions. In four dimensions, there is a sort of uniqueness theorem.⁷³⁾ In five dimensions, the uniqueness of the near-horizon geometry is discussed.⁴¹⁾

§5. Uniqueness of stationary, rotating black holes

5.1. Basic strategy

Now we turn to stationary, rotating vacuum black holes. A critical step toward the uniqueness proof is the reduction of the problem to a boundary value problem of a certain type of non-linear sigma model [recall Step (iii) of section 3]. In 4-dimensions, the rigidity theorem applying a rotational black hole ensures the existence of the 2-dimensional isometry group, $\mathbf{R} \times U(1)$, which is enough to reduce the system to a sigma model on the 2-dimensional factor space $M/\{\mathbf{R} \times U(1)\}$, thereby being decoupled from gravity. In higher dimensions, the rigidity theorem again holds as seen previously, but guarantees the existence of only one rotational symmetry. Therefore, in order to take the same steps as in 4-dimensions, we need to assume multiple axial symmetries, $U(1)^{D-3}$, besides the stationarity \mathbf{R} .

Another important step is to identify boundary conditions on the sigma model fields that are necessary and sufficient to fully determine the solution [recall Step (iv) of section 3]. In 4-dimensions, the desired boundary conditions are given by specifying asymptotic conserved charges, such as the total mass and angular momentum. However, in higher dimensions, these conditions are not enough and we need to specify more data than merely having the same asymptotic conserved charges. We at least need to specify the topology of a black hole considered. As the phase diagram of 5-dimensional stationary rotating vacuum black holes indicates, if we restrict attention to spherical black holes, then the 4-dimensional uniqueness result may be generalized to 5-dimensions. Such a restricted case of uniqueness theorems has been shown for the first time by Morisawa and Ida.⁷⁹⁾ We shall discuss how to prove the uniqueness theorem for topologically restricted, spherical horizon case in the next subsection. But before going into that, let us illustrate basic strategy for the uniqueness proof in more general context.

According to Steps (iii) and (iv), we proceed as follows. (1) We first reduce the D -dimensional Einstein equations with $(D - 2)$ commuting Killing vector fields (one of which is for the stationarity) to a non-linear sigma model, that is, a set of equations for scalar fields on the 2-dimensional factor space B , with the target space isometry G . With the aid of G , the action of the sigma model can be described in terms of a symmetric, unimodular matrix, Φ , on the coset space G/H where H is an isotropy subgroup of G . Thus, the solutions of the original system can be expressed by the matrix Φ . Furthermore, the matrix Φ formally defines a conserved current, Π_i , for the solution. (2) Next, we introduce the *deviation matrix*, Ψ , which is essentially the difference between two coset matrices, say Φ and Φ' , so that when the two solutions coincide with each other, the deviation matrix vanishes, and vice versa; it is usually defined as $\Psi := \Phi'\Phi^{-1} - I$. What we wish to show is that Ψ vanishes over the entire B when the two solutions satisfy the same boundary conditions that specify relevant physical parameters characterising the black hole solution of interest. For this purpose, we construct a global identity, called the *Mazur identity*, (the integral version of) which equates an integration along the boundary ∂B of a derivative of the trace of Ψ to an integration over the whole base space B of the trace of ‘square’ of the deviation, M_i , of the two conserved currents, Π_i and Π'_i . The latter is therefore non-negative. (3) Then, we perform boundary value analysis of the matrix Ψ . We identify boundary conditions for Φ (and Φ') that define physical parameters characterising the corresponding black hole solutions and that guarantee the regularity of the solutions. Then we examine the behavior of Ψ near ∂B . For higher dimensional case, this is the point where we need, other than the asymptotic conserved charges, some additional parameters to fully specify the solutions. Also this is the place where we have to take into consideration the nature of asymptotic structure of the spacetime. When the integral along the boundary ∂B , say the left-side of the Mazur identity, vanishes under our boundary conditions, it then follows from the right-side of the identity, i.e., the non-negative integration over B , that M_i has to vanish, hence the two currents, Π_i and Π'_i , must coincide with each other over B , implying that the deviation matrix Ψ must be constant over B . Then, if Ψ is shown to be zero on some part of the boundary ∂B , it follows that Ψ must be identically zero over the entire B , thus proving the two solutions, Φ and Φ' , must be identical.

5.2. Rotating black holes in 5-dimensions

From now on we focus on the asymptotically weakly simple 5-dimensional spacetimes admitting Abelian isometry group $\mathbf{R} \oplus U(1) \oplus U(1)$. The isometry group is generated by three commuting Killing vector fields U , V and W , where U denotes the generator of the time translation, and each V and W generates spatial rotation with respect to one of the pair of half 2-planes (namely, the symmetric axes), whose orbits are circles. This requirement of isometry is consistent with the asymptotic flatness, for $U(1) \oplus U(1)$ is contained in the rotational group $SO(4) \simeq SU(2) \times SU(2)$ as a subgroup.

The Frobenius conditions

$$\begin{aligned}(\partial_{[a}U_b)U_cV_dW_e] &= 0, \\(\partial_{[a}V_b)U_cV_dW_e] &= 0, \\(\partial_{[a}W_b)U_cV_dW_e] &= 0,\end{aligned}$$

for the integrability of the 2-dimensional distribution orthogonal to the group orbits of the isometry is contained in the vacuum Einstein equation. This can be seen from the identities

$$\begin{aligned}\partial_f[(-g)^{1/2}\epsilon_{abcde}U^aV^bW^c\nabla^dU^e] &= 2(-g)^{1/2}\epsilon_{abcdf}U^aV^bW^cR_e^dU^e, \\ \partial_f[(-g)^{1/2}\epsilon_{abcde}U^aV^bW^c\nabla^dV^e] &= 2(-g)^{1/2}\epsilon_{abcdf}U^aV^bW^cR_e^dV^e, \\ \partial_f[(-g)^{1/2}\epsilon_{abcde}U^aV^bW^c\nabla^dW^e] &= 2(-g)^{1/2}\epsilon_{abcdf}U^aV^bW^cR_e^dW^e.\end{aligned}$$

From the Ricci flatness, the right hand side of each identity becomes zero. This implies that each quantity between the square brackets in the left hand side is constant. However, all these quantities are zero at the fixed points of the symmetric axes, so they must be zero everywhere. This clearly shows the fulfillment of the Frobenius conditions.

The integrability of the 2-distribution orthogonal to the orbits of the isometric actions implies that there is a coordinate system $(x^1, \dots, x^5) = (x, y, t, \phi, \psi)$, where the Lorentzian metric is written in the block diagonal form

$$g = \sum_{i,j=1}^2 h_{ij}(x, y)dx^i dx^j + \sum_{I,J=3}^5 \Phi_{IJ}(x, y)dx^I dx^J,$$

and the Killing vector fields are expressed as

$$U = \partial_t, \quad V = \partial_\phi, \quad W = \partial_\psi. \quad (5.1)$$

We mean the asymptotic flatness by imposing that the metric has an asymptotic form

$$\begin{aligned}g = & - \left[1 - \frac{8GM}{3\pi r^2} + O(r^{-3}) \right] dt^2 + \left[1 + \frac{4GM}{3\pi r^2} + O(r^{-3}) \right] (dx^2 + dy^2 + dz^2 + dw^2) \\ & - \left[\frac{8GJ_\phi}{\pi r^4} + O(r^{-5}) \right] dt(ydx - xdy) - \left[\frac{8GJ_\psi}{\pi r^4} + O(r^{-5}) \right] dt(wdz - zdw), \quad (5.2)\end{aligned}$$

where M , J_ϕ and J_ψ denote the ADM mass and angular momenta, respectively. We further assume that

$$\frac{32GM^3}{27\pi} > (|J_\phi| + |J_\psi|)^2$$

holds.

At this stage, the gravitational fields are described by the 6 scalar fields $\{\Phi_{IJ}(x, y)\}$ and the Riemannian metric $h(x, y)$ on the 2-dimensional base space B^2 parametrized

by (x, y) . More convenient parametrization of $\{\Phi_{IJ}\}$ and h is given by

$$\sum_{I,J=3}^5 \Phi_{IJ} dx^I dx^J = \sum_{I,J=4}^5 f_{IJ} (dx^I + w^I dt)(dx^J + w^J dt) - f^{-1} \rho^2 dt^2, \quad (5.3)$$

$$h = f^{-1} \gamma, \quad (5.4)$$

where $\rho^2 = -\det(\Phi_{IJ}) \geq 0$ and $f = \det(f_{IJ})$. Note that the function ρ^2 becomes zero only on the symmetric axes, which are actually 2-planes though we informally call them axes, and on the event horizon. Furthermore, we introduce the twist potentials ω_I ($I = 4, 5$) by

$$\partial_k \omega_I = \rho^{-1} f f_{IJ} \sqrt{\gamma} \epsilon_{ik} \gamma^{im} \partial_m w^J, \quad (5.5)$$

where ϵ_{ik} is the antisymmetric symbol on the 2-space B^2 specified by $\epsilon_{12} = 1$, and $\gamma = \det(\gamma_{ij})$. The integrability condition of Eq. (5.5) is provided by the Ricci flat condition via

$$\partial_{[i} \partial_{j]} \omega_I = \rho f^{-1} \sqrt{\gamma} R_I^3 \epsilon_{ij}.$$

Hence the twist potential ω_I is globally defined if the 2-dimensional base space B^2 is simply connected as assumed later.

The Ricci flat condition also ensures that the function ρ is a harmonic function on the base space. This can be seen from the identity

$$\Delta^\gamma \rho = \rho^{-1} R_{33} - \rho f^{-1} f^{IJ} R_{IJ}, \quad (5.6)$$

where Δ^γ denotes the Laplace operator on B^2 , and f^{IJ} is the component of the inverse matrix of f_{IJ} . We choose the harmonic function $\rho \geq 0$ and its conjugate harmonic function z as coordinates on B^2 , so that the pair (ρ, z) gives an isothermal coordinate system. [See Ref. 29) for a rigorous proof of the well-definedness of ρ , which is based on the results of Refs. 28), 44). See also earlier work 80).] Hence, the spacetime metric can be written as

$$g = f^{-1} e^{2\sigma} (d\rho^2 + dz^2) - f^{-1} \rho^2 dt^2 + \sum_{I,J=4}^5 f_{IJ} (dx^I + w^I dt)(dx^J + w^J dt),$$

where all the metric functions depend only on ρ and z . This form of the metric is often called the *Weyl-Papapetrou type form*.

Now, the Einstein equation is reduced to the elliptic equations on the (ρ, z) -half plane

$$\rho^{-1} \partial_i (\rho \partial^i f_{IJ}) = f^{KL} (\partial^i f_{IK}) \partial_i f_{JL} - f^{-1} (\partial^i \omega_I) \partial_i \omega_J, \quad (5.7)$$

$$\rho^{-1} \partial_i (\rho \partial^i \omega_I) = f^{-1} (\partial^i f) \partial_i \omega_I + f^{JK} (\partial^i f_{IJ}) \partial_i \omega_K. \quad (5.8)$$

Once these equations are solved for five scalar functions $\{f_{IJ}\}$ and $\{\omega_I\}$, the other metric components are systematically found by integrations of

$$\rho^{-1} \sigma_{,\rho} = \frac{1}{8} f^{-2} [(f_{,\rho})^2 - (f_{,z})^2] + \frac{1}{8} f^{IJ} f^{KL} (f_{IK,\rho} f_{JL,\rho} - f_{IK,z} f_{JL,z})$$

$$\begin{aligned}
& + \frac{1}{4} f^{-1} f^{IJ} (\omega_{I,\rho} \omega_{J,\rho} - \omega_{I,z} \omega_{J,z}), \\
\rho^{-1} \sigma_{,z} &= \frac{1}{4} f^{-2} f_{,\rho} f_{,z} + \frac{1}{4} f^{IJ} f^{KL} f_{IK,\rho} f_{JL,z} + \frac{1}{2} f^{-1} f^{IJ} \omega_{I,\rho} \omega_{J,z}, \\
w_{,\rho}^I &= \rho f^{-1} f^{IJ} \omega_{J,z}, \\
w_{,z}^I &= -\rho f^{-1} f^{IJ} \omega_{J,\rho}.
\end{aligned} \tag{5.9}$$

The Equations (5.7) and (5.8) are derived via the extremization of the action

$$\begin{aligned}
S[\{f_{IJ}\}, \{\omega_I\}] &= \int_{B^2} d\rho dz \rho \left[\frac{1}{4} f^{-2} (\partial^i f) \partial_i f + \frac{1}{4} f^{IJ} f^{KL} (\partial^i f_{IK}) \partial_i f_{JL} \right. \\
&\quad \left. + \frac{1}{2} f^{-1} f^{IJ} (\partial^i \omega_I) \partial_i \omega_J \right].
\end{aligned}$$

This action principle is derived by Maison.⁸¹⁾ In this way, we now have a theory of five real scalar fields on B^2 . Let us see how these five scalars can be embedded in a matrix field in the following. Define a symmetric real matrix field by

$$\Phi = \begin{pmatrix} f^{-1} & -f^{-1} \omega_\phi & -f^{-1} \omega_\psi \\ * & f_{\phi\phi} + f^{-1} \omega_\phi \omega_\phi & f_{\phi\psi} + f^{-1} \omega_\phi \omega_\psi \\ * & * & f_{\psi\psi} + f^{-1} \omega_\psi \omega_\psi \end{pmatrix}.$$

The determinant of this matrix is unity at each point. Clearly, Φ is spectral decomposable at each point, and all the eigenvalues are strictly positive in a point near the spatial infinity. Hence asymptotic flatness ensures that this matrix is strictly positive matrix everywhere. We introduce the equivalence relation in $SL(3, \mathbf{R})$ by

$$g \sim g' \Leftrightarrow \exists Q \in SO(3), \text{ s.t. } g = g' Q.$$

Then Φ uniquely determines $[\Phi^{1/2}] \in SL(3, \mathbf{R})/SO(3)$. Conversely, let $[g] \in SL(3, \mathbf{R})/SO(3)$ be given. When the singular value decomposition of the representative g is given by

$$g = P D Q^{-1},$$

where $P, Q \in SO(3)$ and D is the diagonal matrix composed of singular values of g ,

$$\Phi = P D^2 P^{-1}$$

is clearly a strictly positive symmetric matrix with determinant 1, and it is independent of the choice of the representative. This shows that the matrix field Φ defines the differentiable map $B^2 \rightarrow SL(3, \mathbf{R})/SO(3)$.

Let us define the current matrix field Π_i by

$$\Pi_i = \Phi^{-1} \Phi_{,i}.$$

This belongs to the representation space of the adjoint representation of global $SL(3, \mathbf{R})$ transformation. In terms of this, the elliptic equations (5.7) and (5.8) are equivalent with

$$\partial_i \Pi^i = 0,$$

and this is derived from the extremization of the action

$$S[\{\Pi_i\}] = \frac{1}{4} \int_{B^2} d\rho dz \rho \text{Tr}(\Pi^i \Pi_i).$$

Hence, the problem is reduced to the (weighted) nonlinear σ -model over B^2 with the target space $SL(3, \mathbf{R})/SO(3)$.

Let us assume that there is a single non-degenerate event horizon whose spatial section is diffeomorphic with a 3-sphere. In this case, B^2 becomes (ρ, z) -half 2-plane given by $\rho \geq 0$.

The uniqueness theorem for the nonlinear σ -model on B^2 is obtained utilising either the Mazur identity,⁸²⁾ or the Bunting identity.⁸³⁾

The Mazur identity is in general applied for the nonlinear σ -model with the target space which is a coset space G/H . Let both Φ and Φ' be possibly distinct solutions with the same asymptotic behavior. The Mazur identity is derived from the divergence equation

$$\partial_i [\rho \partial^i \text{Tr}(\Phi' \Phi^{-1} - I)] = \rho \text{Tr}[^t(\Pi^{i'} - \Pi^i) \Phi (\Pi'_i - \Pi_i) \Phi^{-1}], \quad (5.10)$$

where I denotes the identity matrix field. Integrating this equation over B^2 , and applying the Green's theorem, we obtain

$$\oint_{\partial B^2} \rho \partial^i \text{Tr}(\Phi' \Phi^{-1} - I) dS_i = \int_{B^2} \rho \text{Tr}({}^t M^i M_i) d\rho dz, \quad (5.11)$$

where

$$M^i = g^{-1t} (\Pi^{i'} - \Pi^i) g' \quad (5.12)$$

has been defined. The boundary conditions ensures that the left hand side of Eq. (5.11) is zero. However, for the integrand in the right hand side of Eq. (5.11) is nonnegative, it must be zero everywhere. This implies that $M^i = 0$ holds everywhere. It immediately follows that two solutions Φ and Φ' are identical.

Another way to prove the uniqueness theorem is provided by the Bunting method in the following.

$$\partial_i [\rho \partial^i \text{Dist}(x)^2] = \rho G_{AB} h^{ij} (\mathcal{D}_i f_{,s}^A) \mathcal{D}_j f_{,s}^B - \rho^{\mathcal{T}^5} R_{ABCD} f_{,s}^A f_{,i}^B f_{,s}^C f_{,i}^D, \quad (5.13)$$

where

$$\mathcal{D}_{,i} f_{,s}^A = \partial_i f_{,s}^A + \mathcal{T}^5 \Gamma_{BC}^A f_{,i}^B f_{,s}^C.$$

Since the first term on the right hand side of Eq. (5.13) is nonnegative, we have

$$\partial_i [\rho \partial^i \text{Dist}(x)^2] \geq 0, \quad (5.14)$$

if the target space \mathcal{T}^5 has nonpositive sectional curvature. This is true in our case, as explained in the following. The Riemann curvature $\mathcal{T}^5 R^{AB}{}_{CD}$ defines a linear map acting on the 10-dimensional linear space of real antisymmetric tensors, by $V^{[AB]} \rightarrow$

$\mathcal{T}^5 R^{AB}{}_{CD} V^{[CD]}$. This is a normal matrix and has only a triply degenerated nonzero eigenvalue $-5/2$. The nonpositivity of the sectional curvature of \mathcal{T}^5 immediately follows.

Integrating the divergence identity (5.13) over B^2 gives

$$\int_{B^2} \partial_i \rho \partial^i \text{Dist}(x)^2 d\rho dz = \oint_{\partial B^2} \rho \partial_i \text{Dist}(x)^2 dS^i.$$

If the integrand of the right hand side is zero, which, in the present case, is the case by the boundary conditions, the equality

$$\partial_i [\rho \partial^i \text{Dist}(x)^2] = 0 \quad (5.15)$$

should hold. This implies that $\text{Dist}(x) = \text{const}$. However, this constant must be zero, for the asymptotic conditions require $\text{Dist}(x) \rightarrow 0$ ($\rho \rightarrow +\infty$). It immediately follows that $f^A(x) = f'^A(x)$ everywhere. Thus, we have shown the following theorem:

Theorem 5.1: [Reference 79)] *Consider 5-dimensional asymptotically flat, stationary vacuum black holes with two commuting rotational Killing vector fields that commute also with the stationary Killing vector field. Assume further that the solutions have the spherical horizon topology, S^3 . Then, the solution of this system can be uniquely determined by the three asymptotic charges, the mass M and the two angular momenta, J_1 and J_2 , and therefore is isometric to the Myers-Perry metric that has the same values of the corresponding asymptotic charges.*

The above uniqueness result for spherical black holes has been extended to more general cases that include other horizon topologies by Hollands and Yazadjiev³⁹⁾ by employing the interval structure as a set of parameters to completely determine a black hole solution. We quote their theorem:

Theorem 5.2: [Reference 39)] *Consider in 5-dimensions, two stationary, asymptotically flat, vacuum black hole spacetimes with non-degenerate horizon, having two commuting axial Killing fields that commute also with the stationary Killing field. Assume that the both solutions have the same interval structure and the same values of the angular momenta. Then they are isometric.*

Thus, this theorem encompasses black ring solutions^{3), 84)} as well as hypothetical black lens solutions^{*)} (if such a solution exists), besides the spherical black holes.⁵⁵⁾ Note that the mass can be determined by the angular momenta and the interval structure. It is worth emphasising that the interval structure can completely determine not only the topology of each connected component of the event horizon but also the topology of the domain of outer communications, as well as the action of the rotational isometries. The above theorem has been further generalized to include asymptotically Kaluza-Klein black holes.⁴⁰⁾

^{*)} Some attempts to find an exact black lens solution have been made in Ref. 85) [see also, e.g., Ref. 86)]. However, the ‘black lens solutions’ so far constructed turned out to suffer from a naked singularity.

§6. Summary

We have seen basic properties of black holes in higher dimensions, focusing mainly on the subjects relevant to the uniqueness theorems for asymptotically flat, stationary vacuum solutions in general relativity. After a brief summary of the global aspects relevant to the black hole area theorem in higher dimensions, we have seen rich topological aspects of dynamical, non-stationary black holes^{22), 24)} in section 2, where the crease set plays a key role. Galloway and Schoen's topology theorem¹⁶⁾ provides the non-trivial restriction on possible topology of apparent as well as the event horizons. When a black hole spacetime admits axial/rotational symmetries, we have topological restrictions stronger than that of Galloway and Schoen as seen in section 3.1.1. This is relevant in particular when we are concerned with stationary black holes since 'stationary' implies 'axisymmetric' due to the rigidity theorem. In particular, when a stationary black hole admits multiple rotational symmetries so that the factor space of the black hole spacetime by its isometry reduces to 2-dimensional, the rod/interval structure^{36), 39)} can completely specify the horizon topology as briefly commented in section 3.1.1 and at the end of section 5.2. When there is only a single rotational symmetry in 5-dimensions, we have a weaker topological restriction,⁴²⁾ which allows for the possibility of 'Black Prism' and other various Seifert manifolds as the horizon cross section manifold, though such a larger variety of the horizon manifold than those already known in the existing exact solutions may possibly be ruled out once the Einstein equations are fully taken into account. In section 3.2, we have seen the main idea of the rigidity proof in higher dimensions,⁵⁰⁾ and its extension to extremal black holes with some additional technical assumption.⁵⁶⁾ It guarantees the existence of only one rotational symmetry, which is in accord with the conjecture of existence of less symmetric stationary black holes,⁴¹⁾ while all known exact solutions have multiple rotational symmetries. For this reason, in order to obtain some uniqueness results by following basic steps described in section 5.1—similar to Steps (i)–(iv) of the 4-dimensional case, one has to impose multiple $(D - 3)$ -axial symmetries, besides the assumed stationary symmetry. By doing so, one can prove the uniqueness of stationary rotating black holes in higher dimensions with some additional data, such as a spherical topology⁷⁹⁾ or more generally the rod/interval structure,³⁹⁾ other than the asymptotic conserved charges as seen in section 5.2. For the static case, the existing proof in 4-dimensions uses some geometric properties that hold only in 4-dimensions, and therefore it is not straightforward to generalize to higher dimensions. We have seen how to overcome this difficulty^{47), 49)} in section 4.

The uniqueness results in higher dimensions discussed in section 5 hold only for non-extremal black holes. In 4-dimensions, generalizations to uniqueness theorems for extremal Kerr and extremal charged Kerr black holes have recently been made by Refs. 87), 88), 89). The key new element for the proof is the uniqueness of the near-horizon geometry for a degenerate horizon of stationary axisymmetric vacuum spacetimes,^{90), 91), 92)} i.e., the near-horizon geometry must agree with that of the extremal Kerr metric,⁹³⁾ and a similar result also holds for the extremal electrovacuum black hole case. In 5-dimensions, however, near horizon geometries for stationary

vacuum solutions with two rotational symmetries are not unique. [See Ref. 92) for the lack of the uniqueness and Refs. 92), 94), 95), 96), 97) for classification of near-horizon geometries.] Therefore it seems to be highly non-trivial whether one can obtain similar uniqueness results for extremal black holes in higher dimensions. In this regard, it is interesting to note Theorem 2 of Ref. 89) that the interval structure can be used to uniquely determine extremal vacuum black holes in 5-dimensions, (as well as near-horizon geometries) under similar assumptions of the uniqueness theorem 5.2 above.³⁹⁾

It turns out that on one hand, the rod/interval structure provides a convenient set of data to uniquely determine a stationary black hole spacetime with $(D - 3)$ rotational symmetries. It may be possible to completely classify all stationary vacuum black hole solutions, using the rod/interval structure. On the other hand, as it is a local notion, the use of the rod/interval structure in uniqueness theorems does not appear to be fully satisfactory; an asymptotically flat black hole, as an isolated system, should be described in terms of asymptotic data as accurate as possible. However, as indicated by the existence of multiple black objects such as black Saturn⁹⁸⁾ and di-ring,⁹⁹⁾ if attempting to completely characterize a black hole solution by using asymptotic data (e.g., multipole moments defined at infinity) alone, then one would need to identify an infinite set of the multipole moments. In view of this, it would be interesting to clarify relations between the rod structure and asymptotic conserved charges and multipole moments. [See Ref. 100) for recent progress along this line.] The task would be to identify a minimal set of local data (e.g., the number of connected components of the event horizon) and a maximal finite set of asymptotic data.

Acknowledgments

AI is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS (A)No. 22244030 (C)No. 22540299. TS is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 21244033, No. 21111006, No. 20540258 and 19GS0219.

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